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APPROXIMATE PREDICTION OF MULTIMODAL CREST STATISTICS AND SYSTEM RELIABILITY FOR IMPULSIVE NOISE LOADING

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FOREWORD

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ABSTRACT

The current analytical models of impulsive noise and system response thereto are found numerically intractable, indicating a need for engineering approximations. One, based on finding an "equivalent" unimodal system having approximately the same response statistics--here the crest statistics--is examined. It is found that fair agreement obtains, making it possible to use two reasonably accessible parameters (γ and ω_0) to characterize any linear multimodal system response and thus predict its crest statistics systematically, provided that "standard" curves of unimodal response crest statistics are available for statistically identical forcing. Limitations of this promising approach are explored, encouraging its use in reliability prediction pending further studies. Also, the need for further studies of impulsive noise response crest distribution "tails" is noted for prediction of overload and wearout reliability on the basis of wear-dependent failure rate concepts, since extrapolation of empirical data into the "tail" (here done on the basis of a Weibull crest fit) is increasingly necessary the more "impulsive" the response is (i.e., the more the parameter γ is less than unity).

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INTRODUCTION

The prediction of system reliability under threat of overload and wearout failure is currently performed in terms of system response "peak" (crest, maxima, or rise) statistics for the standard models of these two basic failure mechanisms. Hence, one important problem confronting the systems analyst is to find these "peak" statistics--particularly the level crossing (crest) statistics which yield conservative estimates--for the system in question under an anticipated random load.

Due to the prevalence of nongaussian, particularly impulsive (characterized by large-amplitude bursts against a low-amplitude background) noise in realistic models of random load environments, the analytical difficulties are formidable in an accurate computation of the pertinent "peak" statistical distributions. Hence, one would like to learn how to apply current analytical models for statistical system response to impulsive noise in order to estimate "peak" statistics. Moreover, one desires "engineering" response models which will serve, in particular physical situations, until more general analytical results have been found. Thus, it is of more than academic interest to learn both the limitations and potentialities of the results obtained for the impulsive-noise-response of a simple linear harmonic oscillator, mutatis mutandis, to predict the "peak" statistics (and thus reliability) of more complex multimodal linear systems.

This report will first briefly summarize the state of the art in the analysis of impulsive noise response of linear, time-invariant systems. Then, possibilities of finding approximately "equivalent" unimodal systems--facilitating reliability prediction for complex multimodal systems--will be explored. It will be seen that multimodal system "peak" statistics may often be conservatively (but not excessively so) estimated for reliability predictions, the system being characterized by two physically meaningful parameters, a "time constant" τ (or bandwidth) and an "equivalent resonant frequency" ω_o.

It was found that utilizing initial-slope decay times yielded lower bound estimates for the empirical crest distributions, while the bandwidth parameter yielded conservative (upper bound) estimates. Predictions of the exact frequency of zero-crossings (necessary for normalization), as well as the equivalent unimodal system resonant frequency, presented certain difficulties; but, one obtains normalized crest distribution predictions (frequency of level-crossings divided by that of zero-crossings) that are above the corresponding unimodal statistics by a factor of the order of unity. Potentialities and limitations of these models are elucidated, and areas of further research delineated in subsequent sections of this report.

II

STATISTICAL MODELS FOR SOURCES AND SYSTEMS

Reliability predictions for a complex system under random loading generally assume knowledge of a relation between some particular aspect of the system's response and system failure (ref. 1). With this information one then usually attempts to predict the average time to failure. For this purpose the system response "peak" distributions (crests, maxima, or rises) are typically utilized in overload and wearout reliability calculations. Previous workers have found that, insofar as partial wear (damage) predictions are concerned, a hierarchy exists among the "peak" statistics (refs. 2, 3). For a given situation the crest statistics offer the most conservative (i.e., shortest) estimate of the time to failure, while the rise statistics provide the most optimistic predictions. For the purposes of this report the "conservative" crest statistics will be the main statistical distribution of interest. The crest statistics, of course, are also of particular interest in overload failure prediction (refs. 4, 5).

The response "peak" statistics may be approached from either analytical or empirical viewpoints. For analytical predictions one must have sufficient "higher order" statistical information (cf. Section III) about the noise source as well as a characterization of the system under study so that formal expressions for the system response statistics can be obtained. This approach, if successful, permits a systematic study of alterations in the average time to system failure precipitated by changes in either the noise source parameters or the system parameters. However, in most physical situations, particularly nongaussian noise loading, when such detailed information is not available, one must resort to fitting empirically the system response "peak" distributions by analytical curves (say Weibull (ref. 4, 5), exponential, Gamma, etc.). At best, one might observe the development of trends which could be linked directly to parameter changes either in the noise source or in the system itself. For the sake of general applicability the analytical approach outlined above will be attempted as extensively as possible. However, when an impasse is reached, the second method will be utilized without further justification.

1. IMPULSIVE NOISE MODELS

In an attempt to analytically characterize environmental impulse noise, various models have been proposed and have received study in the literature (refs. 6-9). Generally speaking, these models possess a two-fold purpose. First, they strive to describe as accurately as possible the environmental noise. Implicit in this operation is the establishment or the definition of the parameters which are deemed crucial in describing the physical noise process. Second, the models are formulated so as to facilitate theoretical predictions of the desired noise statistics. Presumably, then, the final analytical results (e.g., system response crest or maxima statistics or the response first-order probability density) are explicitly dependent on the parameters chosen to describe the noise process.

When analyzing a system forced by impulse noise one has two alternatives. Either the actual noise source or the response of the system itself may be related to an impulse noise model. With the first approach, i.e., noise source modeling, one is still concerned with calculating the response of some system coupled to that source. Thus, it is also necessary that the system be characterized. Typically what is needed is some "input-output" transfer information of the system to account for its response to the given noise source. Although this approach may lead to more complex analytical operations, it does subdivide the pertinent environmental parameters into two groups, one associated with the noise source and the other group related to the system itself. In this fashion the system response can be directly related to changes in the noise source or to changes in the system. The other alternative, that of modeling the actual system response directly, may lead to easier analytical analysis. But at the same time, one loses the ability to isolate any "cause-effect" relationships which may exist between the response statistics and the individual parameters of the noise source and of the system.

Two analytical models are used below. The first, the Poisson impulse noise model, is used here to model the actual noise source (and, mutatis mutandis, the response). The second approach, Hall's generalized-t model, must be applied directly to the system <u>response</u> under impulse loading.

a. Poisson Impulsive Noise Model

S. O. Rice's Poisson shot noise model (refs. 6, 8, 9) offers the greatest promise insofar as analytical elegance is concerned while at the same time elucidating the physically meaningful parameters. Mathematically, this model represents the forcing noise y(t) as a "train" of impulse functions:

$$y(t) = \sum_i a_i \, \delta(t-t_i) \quad . \tag{1}$$

The occurrence times $\{t_i\}$ are assumed i) to be statistically independent of the impulse strengths $\{a_i\}$, and ii) to occur in a Poisson fashion[1] with average impulse rate (or frequency) ν. The $\{a_i\}$, in turn, are assumed to be mutually independent random variables with a common impulse strength probability density $p_a(a)$. Unfortunately, the resultant formal quadrature or series expression for say the response first-order probability density or crest statistics of a linear system forced by Poisson noise are usually not readily converted into numerical results (refs. 10, 11) suitable for reliability predictions. Typically, formal results contain either characteristic functions which cannot be readily inverse-transformed or infinite summations of Hermite polynomials whose convergence is either extremely slow or even entirely in doubt. Thus, utilization of the Poisson model which is physically appealing, is somewhat limited pending advances in numerical computational techniques.

b. Generalized-t Impulsive Noise Model

The generalized-t model reported in a recent work (ref. 7) by H. M. Hall also offers promise for characterization of impulse noise. Actually, Hall was primarily concerned with modeling radio noise caused by atmospheric disturbances (e.g., lightning) as observed at the output of a communications receiver. Thus, his results are most directly applicable to filtered "impulsive" atmospheric radio noise. Hall's model

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[1]The Poisson distribution giving the probability of n impulses occurring in an observation time T is given by

$$P_T(n) = \frac{(\nu T)^n}{n!} e^{-\nu T} \quad .$$

is therefore of the second type outlined above, i.e., one associated directly with the system response process. From a mathematical point of view Hall's model is applicable without major modification only to the output response of the system being forced by impulsive noise.

A formulation of the generalized-t model proceeds as follows. Let the response process y(t) be represented as a product of two statistically independent noise processes denoted by m(t) and n(t), i.e.,

$$y(t) = m(t)n(t) \quad . \tag{2}$$

Hall's analysis required that the "carrier" n(t) be an ensemble member of a narrowband Guassian process with correlation function $R_n(\tau)$, while the "modulating" process m(t) be a slowly varying (with respect to n(t)) process with a known first-order probability density function, $p_m(m)$. Actually, n(t) need not be restricted to a narrow-band process. It turns out that a necessary requirement is that the frequency spectra of n(t) and m(t) be non-overlapping.

Using the above assumptions about m(t) and n(t) one can formally express various densities of y(t) as integrals (i.e., averages) of known Gaussian densities. For example, one immediate result[2] is that the first-order probability density of y(t), $p_y(y)$, be given by

$$p_y(y) = \int_{-\infty}^{\infty} \frac{dx}{|x|} \, p_n\left(\frac{y}{x}\right) p_m(x)\, dx \quad . \tag{3}$$

Since n(t) is a member of a Gaussian process, the form of $p_n(\cdot)$ is known. Thus, specification of $p_m(\cdot)$ formally defines $p_y(y)$. Of course, analytical evaluation of the above integral may become an obstacle. Other statistical distributions, for example the crest or maximum statistics of y(t), also can be obtained, at least in quadrature; such problems will receive further attention in Section III. Thus, the formal analytical simplicity of Hall's impulse noise model is appealing,

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[2] The requirement that m(t) be "slowly varying" is not necessary in order to obtain this particular result for $p_y(y)$, but this restriction is quite important in computing "peak" statistics.

although its suitability for engineering predictions is still a point in question.

Unfortunately, a logical association between the pertinent parameters of Hall's model and available environmental data is quite obscure. In contrast, one of the attributes of the Poisson model is the rather obvious correspondence between the noise model impulse rate ν and the observed rate of occurrence of noise pulses from field data (ref. 12). In his work, Hall modeled response envelope data obtained by filtering atmospheric radio noise. He found that a chi-squared distribution for the random variable $b \triangleq 1/m$,

$$p_b(b) = \frac{\left(\frac{M}{2}\right)^{M/2}}{\sigma^M \Gamma\left(\frac{M}{2}\right)} |b|^{M-1} \exp\left\{-\frac{M}{2\sigma^2} b^2\right\} , \tag{4}$$

yields a proper "asymptotic behavior" of the first-order probability density $p_y(y)$ for large y as suggested by other investigators (ref. 13). Then, by adjusting the two parameters in Eq. (4) namely (M and σ), one is able to obtain "best fits" to filtered atmospheric radio noise envelope data (first-order envelope probability densities and envelope level crossing rates). The selection of appropriate model parameters is, then, based solely on data observed at the output of the system in question. Thus, although one may be able to model the system response, it is not at all obvious how these analytical response statistics are related to certain parameters of either the noise source (e.g., average impulse occurrence rate) or of the system itself (e.g., the system bandwidth).

One faces comparable problems when endeavoring to obtain damage estimates for any physical situation when employing the generalized-t model to characterize system response to impulsive noise environments. Moreover, the use of this model in damage calculations is misleading in that the formal results may produce infinite damage moments (cf. Eq. (30)). Nevertheless, on consideration of physically useful models for impulsive noise, the above mentioned limitations of the generalized-t model certainly are not so formidable as to rule out the method, mutatis mutandis.

2. SYSTEM MODELING

When using the Poisson model of the preceding section, one must necessarily characterize analytically the linear system under study. Even though the generalized-t model is applicable only to the system response, one would suspect that a detailed characterization of the linear system might possibly be directly associated with resultant changes in the response statistics. Thus, even with Hall's modeling procedure, system modeling should prove quite useful.

None of the preceding discussions have focused directly on the form of system characterization to be used. The formulation of the Poisson noise model in terms of "impulse functions" suggests strongly that the system impulse response function $h(\tau)$ should be embodied in the system modeling process. Once a $h(\tau)$ is specified, the system in question (assumed in this report to be linear and time-invariant) is fully described (ref. 14). For a complex system, however, one may not be able to obtain $h(\tau)$ in a tractable form. In fact only for rather simple systems can $h(\tau)$ be readily expressed in closed form.

For our purposes perhaps the most common system for which this characterization is possible is the simple linear harmonic oscillator. It is a natural building block for more complex linear systems; of particular interest here are underdamped (high Q) systems. Such a system has an impulse response (ref. 15) given by

$$h(\tau) = A\, e^{-\frac{\omega_o}{2Q}\tau} \sin\omega_o\tau \qquad (5)$$

where ω_o is the undamped resonant frequency, Q is the system quality factor, and A is a normalization ("gain") factor. In this form, Eq. (5) is particularly amenable to analytical calculations and predictions of system reliability.

Characterization of more complex systems by the specification of $h(\tau)$ is not readily accomplished. For example, a two-degree-of-freedom system which can be described by a pair of coupled second-order linear differential equations (with constant coefficients) such as Eq. (46) in Section V yields an impulse response of the form

$$h(\tau) = A_1 \epsilon^{-K_1 \tau} \sin(\omega_1 \tau) + B_1 \epsilon^{-K_2 \tau} \sin(\omega_2 \tau) \qquad (6)$$

for certain choices of system parameters. It should be observed that Eq. (6) basically does involve linear combinations of the more simple $h(\tau)$ characteristics of the single-mode case, Eq. (5). However, the analytical form of Eq. (6) is still sufficiently complex to obviate convenient utilization for predicting system response statistics to impulsive loading. If one considers higher-degree-of-freedom systems, the situation is further obscured and for a distributed-parameter system, the expressions for $h(\tau)$ are available only as infinite sums over the eigenmodes of the system (cf. discussion of a clamped circular plate in the Appendix).

Faced with these inherent difficulties one is forced to investigate as to whether or not less information than the complete specification of $h(\tau)$ would suffice for making damage predictions. As an example, one might ask if some "average system response time" information is sufficient; or possibly an "equivalent single-degree-of-freedom system" could be found and used as a statistically reliable model for the more complex system when making the predictions. These specific questions and others will be discussed later in more detail. It will be shown that for certain engineering estimates one can extend known unimodal system behavior to new situations involving complex systems utilizing only a minimal amount of information about the system in question.

III

"EXACT THEORIES" FOR PREDICTING "PEAK" DISTRIBUTIONS

The response "peak" distributions under consideration formally can be expressed in terms of higher-order response joint distributions. More often than not, however, calculations based on these formal solutions cannot be converted into numerical results. In some cases computational difficulties can be circumvented by using approximations valid in the low-impulse frequency limit as discussed in sub-section 1. In sub-section 2 "exact" results are derived for the crest and maxima statistics employing the generalized-t response model. Although the analytical results are relatively simple in form, their utilization in damage prediction is still a point in question; no relationships between the model parameters and field parameters have yet been established.

1. POISSON IMPULSIVE NOISE--LOW FREQUENCY LIMITS

As stated earlier formal expressions for the system response crest, maxima and rise distributions are known (refs. 6, 16). Results for the crest and maxima statistics follow from a route outlined by S. O. Rice (ref. 6), namely, that for a stationary random process y(t) the positive-slope level crossing rate of a fixed level y_o, $\nu_c^+(y_o)$, is given by

$$\nu_c^+(y_o) = \int_0^\infty \dot{y}\, p_{y,\dot{y}}(y_o,\dot{y})\, d\dot{y} \, , \tag{7}$$

while the maxima frequency probability density $m^+(y_o)$ can be expressed as

$$m^+(y_o) = \int_{-\infty}^{0} |\ddot{y}|\, p_{y,\dot{y},\ddot{y}}(y_o,0,\ddot{y})\, d\ddot{y} \, . \tag{8}$$

In the expressions Eqs. (7) and (8), $p_{y,\dot{y}}(\cdot,\cdot)$ and $p_{y,\dot{y},\ddot{y}}(\cdot,\cdot,\cdot)$ represent the simultaneous joint probability densities of the response with its first derivative and of the response with its first and second derivative, respectively. The formal expression for the "rise" distribution (refs. 1, 16) involves the "method of inclusion and exclusion" leading to an untractable series representation.

Formally, the crest and maximum distributions can be expressed as integrals of doubly-infinite and triply-infinite series of generalized Hermite polynomials (ref. 17) with the expansion coefficients related in known form (for Poisson impulsive noise response) to the noise source parameters ($\nu, p_a(a)$) and also to the system impulse response $h(\tau)$ (ref. 18). Unfortunately, earlier work has indicated (refs. 10, 11) that for the cases of interest, even a singly-infinite Hermite expansion (for the response first-order probability density) does not converge rapidly enough to allow numerical computational techniques to produce usable results. One would expect even greater difficulty with the more complex expansions arising from Eq. (7) and Eq. (8), even after the integration has been formally performed. Meaningful results for the "peak" statistics can be obtained only for the highly nongaussian, "low ν" limiting cases which are of considerable interest. Two such approaches are illustrated below.

The first approach has been reported earlier (refs. 10, 11, 19) and will be described here very briefly. If one assumes Poisson impulsive noise excitation of a linear single-degree-of-freedom (1DOF) system with a fairly high quality factor (say $Q \gtrsim 5.0$), and if the noise source impulse rate ν is sufficiently small ($\nu \lesssim \omega_o/Q$), then the normalized crest statistics are given by (refs. 10, 19)

$$c^+(\eta) = 4\gamma \int_{\sqrt{\frac{\gamma}{2}}}^{\infty} da \; p_a(a) \; \log \frac{a/2}{\eta\sqrt{\gamma}} \; . \tag{9}$$

Here, η is the normalized (with respect to the rms output) response variable, $p_a(a)$ is the noise impulse strength density, and γ is the impulse-noise-density parameter defined by

$$\gamma \equiv \frac{Q\nu}{\omega_o} \; . \tag{10}$$

Comparison with experiment has shown (ref. 11) that Eq. (9) is valid for $\gamma \lesssim 0.5$. This prediction, Eq. (9), for the crest statistics is valid for high-Q unimodal systems. One would like to extend this approach, based on observations of the system's impulse response (here a single damped-sinusoid)

in order to obtain appropriate limiting forms of the response crest and maximum distributions for more complex systems.

In principle, this approach to predicting response peak statistics in the low impulse frequency limit is applicable to all time-invariant linear systems and (inter alia) to all three types of "peak" statistics; the crests, maxima and the rises (ref. 3). It calls for detailed knowledge of certain aspects of the system's impulse response function $h(\tau)$. The response of a time-invariant linear system to the Poisson noise of Eq. (1) is:

$$y(t) = \sum_i a_i h(t-t_i) \quad . \tag{11}$$

Assuming that $h(\tau)$ decays essentially to zero before some time T, and requiring $\nu << 1/T$ Eq. (11) becomes a sequence of essentially nonoverlapping impulse responses with independent, random scale factors a_i. Thus, the average exceedance frequency of peaks of type p (p = crests, maxima or rises) above the level y_o, $\nu_p(y_o)$, is merely the frequency of these peaks in a single impulse response with the scale factor, a, appropriately averaged over its distribution, i.e.,

$$\nu_p(y_o) = \int_{-\infty}^{\infty} \frac{N_p(y_o,a)}{1/\nu} \, p_a(a)da \quad , \tag{12}$$

where $N_p(y_o,a)$ denotes the integral number of peaks above the level y_o in an impulse response of scale factor a. The peak exceedance probability ("peaks" being by convention positive) is just Eq. (12) normalized to unity at $y_o = 0$, $\nu_p(y_o)/\nu_p(0)$. The only task remaining then is the detailed evaluation of $N_p(y_o,a)$.

The crest distribution, being of prime concern here, will be considered in detail. Figure 1 illustrates the impulse response of strength $1 > 0$ for a typical complex oscillatory system. It can be seen from this figure that for y_o in any of the ranges $aM_{i+1} < y_o < aM_i$ (or, equivalently, for $y_o/M_i < a < y_o/M_{i+1}$) the number of positive-slope crossings of the level y_o, $N_c(y_o,a)$, is constant and equal to the difference between the number of maxima and the number of minima above y_o; similar arguments apply for $a < 0$. Thus, Eq. (12) can be decomposed into a sum of integrals over the

ranges of "a" for which $N_c(y_o, a)$ is constant. Denoting the crest distribution by $c^+(y_o)$ we have:

$$c^+(y_o) = \frac{\nu}{\nu_c(0)} \sum_i N_i \int_{y_o/M_i}^{y_o/M_{i\pm 1}} p_a(a)\,da \quad . \tag{13}$$

In this expression N_i is the number of maxima minus the number of minima in $h(\tau)$ greater than or equal to M_i for $i > 0$ and less than or equal to M_i for $i < 0$. Also, the upper sign in the $\pm$ corresponds to $i > 0$ and the lower sign to $i < 0$. Breaking each integral up into the sum of two semi-infinite integrals and using the normal distribution for $p_a(a)$[3] yields:

$$c^+(y_o) = \frac{\nu}{2\nu_c(0)} \left\{ \sum_i \pm N_i \frac{2}{\sqrt{\pi}} \int_{\xi_i}^{\infty} \exp(-u^2)\,du - \sum_i \pm N_i \frac{2}{\sqrt{\pi}} \int_{\xi_{i\pm 1}}^{\infty} \exp(-u^2)\,du \right\} \tag{14}$$

where $\xi_i \equiv \dfrac{y_o}{\sqrt{2}\,\sigma_a |M_i|}$.

A change in the index of summation in the second sum produces:

$$c^+(y_o) = \frac{\nu}{2\nu_c(0)} \sum_i \pm (N_i - N_{i\pm 1}) \operatorname{erfc} \xi_i \tag{15}$$

where the complementary error function has its standard definition (ref. 20):

$$\operatorname{erfc}(x) \equiv \frac{2}{\sqrt{\pi}} \int_x^{\infty} \exp(-u^2)\,du \quad . \tag{16}$$

Now note that $\pm(N_i - N_{i\pm 1})$ is plus one when M_i is a positive maximum or negative minimum and minus one if M_i is a positive minimum or negative maximum. This leads to

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[3] The argument to be presented is obviously valid for other impulse strength distributions; however, use of the normal distribution for $p_a(a)$ provides a good statistical model for many environmental noises and has been extensively used in empirical studies including those here reported.

$$c^{+}(y_o) = \frac{\nu}{2\nu_c(0)} \left\{ \sum_{\substack{\text{pos max}\\ \text{neg min}}} \operatorname{erfc}\xi_i - \sum_{\substack{\text{neg max}\\ \text{pos min}}} \operatorname{erfc}\xi_i \right\}. \quad (17)$$

It is customary in data-reduction to normalize the level y_o with respect to the rms of the response process, σ_y. This rms level follows by an application of Campbell's theorem (ref. 6):

$$\sigma_y^{\,2} = \nu\sigma_a^{\,2} \int_0^\infty h^2(t)\,dt. \quad (18)$$

Thus the normalized crest distribution becomes:

$$c^{+}(\eta) =$$

$$\frac{\nu}{2\nu_c(0)} \left\{ \sum_{\substack{\text{pos max}\\ \text{neg min}}} \operatorname{erfc}\left[\left(\frac{\nu}{2}\int_0^\infty h^2(t)\,dt\right)^{1/2} \frac{\eta}{|M_i|}\right] - \sum_{\substack{\text{neg max}\\ \text{pos min}}} \operatorname{erfc}\left[\left(\frac{\nu}{2}\int_0^\infty h^2(t)\,dt\right)^{1/2} \frac{\eta}{|M_i|}\right] \right\} \quad (19)$$

where $\eta \equiv \dfrac{y_o}{\sigma_y}$.

Similar computations for the positive maxima and rises (ref. 3) yield their respective exceedance probabilities:

$$M^{+}(\eta) = \frac{\nu}{2\nu_M(0)} \sum_{\substack{\text{pos max}\\ \text{neg min}}} \operatorname{erfc}\left[\left(\frac{\nu}{2}\int_0^\infty h^2(t)\,dt\right)^{1/2} \frac{\eta}{|M_i|}\right]; \quad (20)$$

$$R(\eta) = \frac{\nu}{2\nu_R(0)} \sum_{i} \operatorname{erfc}\left[\left(2\nu \int_0^\infty h^2(t)\,dt\right)^{1/2} \frac{\eta}{|M_{i+1}-M_i|}\right]. \quad (21)$$

Obviously the method outlined above is restricted to the class of rather elementary systems whose impulse response functions [or at least the extrema of $h(\tau)$ and $\int_0^\infty h^2(t)\,dt$] are sufficiently known or can be empirically determined. This limitation is overcome for certain engineering predictions

in Section IV where the results predicted by Eq. (9), or those obtained by specializing Eqs. (19)-(21) to the unimodal system of Eq. (5), are used to (approximately) predict response statistics for more general systems.

Before concluding this discussion, the above mentioned results of Eqs. (19)-(21) for the 1 DOF system will be presented. For the unimodal impulse response of Eq. (5) the maxima and minima are easily calculated yielding for the 1 DOF peak statistics:

$$C^{+}(\eta) = M^{+}(\eta) = \frac{\pi\nu}{\omega_o} \sum_{n=1}^{2\left[\frac{\omega_o}{2\pi\nu}\right]} \operatorname{erfc}\left\{\frac{1}{2}\, \gamma^{1/2} \exp\left(\frac{\pi}{4Q}(2n-1) - \frac{1}{4Q^2}\right)\eta\right\} \tag{22}$$

and

$$\begin{aligned} R(\eta) &= \frac{\pi\nu}{\omega_o} \operatorname{erfc}\left\{\gamma^{1/2} \exp\left(\frac{\pi}{4Q} - \frac{1}{4Q^2}\right)\eta\right\} \\ &+ \frac{\pi\nu}{\omega_o} \sum_{n=1}^{2\left[\frac{\omega_o}{2\pi\nu}\right]-1} \operatorname{erfc}\left\{\frac{1}{2}\, \gamma^{1/2} \exp\left(\frac{n\pi}{2Q} - \frac{1}{4Q^2}\right) \operatorname{sech}\left(\frac{\pi}{4Q}\right)\eta\right\} \end{aligned} \tag{23}$$

with $\left[x\right]$ the greatest integer function $x-1 < \left[x\right] \leq x$ and γ is defined in Eq. (10). Although it is not immediately obvious from the form of these expressions, for large Q the statistics depend only on the impulsive-noise-density parameter γ while exhibiting negligible dependence on Q, ν, or ω_o individually. This agrees with the result of Eq. (9) derived on the basis of a high system Q. The physical significance of the system parameter ω_o/Q will be discussed in Section IV.

2. GENERALIZED-t IMPULSIVE NOISE MODEL

The basic hypotheses of Hall's "generalized-t" impulse noise model in a formal way facilitate the calculation of both crest and maximum statistics. Although the physical impli-cations of the final results caution against using Hall's model to characterize system wear data one can nevertheless appreciate its analytical tractability. For the present

discussion, the most significant assumption is that the modulating process m(t) be slowly-varying when compared to n(t).[4] As will be shown, this hypothesis is most crucial.

Consider, then, the process characterized in Eq. (2). Differentiating Eq. (2) twice yields

$$\dot{y}(t) = m(t)\dot{n}(t) + \dot{m}(t)n(t) \tag{24}$$

and

$$\ddot{y}(t) = m(t)\ddot{n}(t) + 2\dot{m}(t)\dot{n}(t) + \ddot{m}(t)n(t) \quad . \tag{25}$$

If m(t) varies <u>slowly</u> with respect to n(t), then only the first term of the right-hand members of Eqs. (24) and (25) need be retained (ref. 7). One can use this simplification to evaluate Eq. (7). The density function

$$p_{y,\dot{y}}(y,\dot{y}) = \int_{-\infty}^{\infty} \frac{dm}{m^2}\, p_{n,\dot{n},m}\left(\frac{y}{m}, \frac{\dot{y}}{m}, m\right) \tag{26}$$

follows from a change of variables in $p_{n,\dot{n},m}(\cdot,\cdot,\cdot)$ to obtain $p_{mn,m\dot{n},m}(\cdot,\cdot,\cdot) \cong p_{y,\dot{y},m}(\cdot,\cdot,\cdot)$ and then integrating over all values of m. Since n(t) is Gaussian and is independent of m(t), one obtains

$$p_{n,\dot{n},m}(n,\dot{n},m) = \frac{p_m(m)}{2\pi\sigma_n\sigma_{\dot{n}}} \exp\left\{-\frac{n^2}{2\sigma_n^{\,2}} - \frac{\dot{n}^2}{2\sigma_{\dot{n}}^{\,2}}\right\} \tag{27}$$

where (assuming n(t) is zero-mean)

$$\sigma_n^{\,2} = R_n(0)$$
$$\sigma_{\dot{n}}^{\,2} = -\left.\frac{d^2R_n(\tau)}{d\tau^2}\right|_{\tau=0} \quad . \tag{28}$$

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[4] Otherwise the "peak" statistics get rather complicated. After all, any process can be represented as a Gaussian n(t) times some "finagle factor" process m(t).

Combining Eqs. (7), (26), and (27) and performing the integration over the variable $\dot{y}$ leads to the normalized crest statistics

$$c^{+}(y_o) = \int_{-\infty}^{\infty} dm\, p_m(m)\, e^{-\frac{y_o^2}{2\sigma_n^2 m^2}} . \tag{29}$$

Thus, the crest statistics for Hall's model are obtained in quadrature. It is also evident that appropriate choice of $p_m(\cdot)$ may lead to closed-form evaluation of Eq. (29).

An approach similar to that used for the derivation of Eq. (29) also can be employed to derive an expression for the maximum statistics. Using the "slowly varying" assumption on m(t) again simplifies Eqs. (24) and (25), allowing the density function $p_{y,\dot{y},\ddot{y}}(\cdot,\cdot,\cdot)$ to be obtained from $p_{n,\dot{n},\ddot{n},m}(\cdot,\cdot,\cdot,\cdot)$ in a fashion similar to that used for the crest statistics. The general result for the maximum statistics is quite formidable unless a narrowband assumption is made for n(t). For this case one finds that the maxima exceedance distribution $M^{+}(y_o)$ is identical to $c^{+}(y_o)$ and thus is given by Eq. (29). If n(t) is not narrowband, then $M^{+}(y_o)$ is more complicated. Nevertheless, it is in quadrature and involves explicitly only the exponential and the error function. The density function $p_m(\cdot)$ appears within the integrand as a multiplier, just as in Eq. (29). Since the maximum statistics of a multimodal linear system are not examined experimentally in this report, the main point of this discussion is primarily to illustrate the analytical tractability of Hall's model.

The results of Eq. (29) and the discussion above of the response maximum statistics may be obtained in another "intuitive" fashion. One might envision the product representation of Eq. (2) as a "scaling" of the Gaussian variate n(t) by an independent slowly-varying "scale" factor m(t). Thus, the results for both the crest and maximum statistics of y(t) can be obtained by averaging with weight $p_m(m)$ both the crest and maximum statistics of n(t) over the values of m(t), which may be regarded formally as variations in the rms level of n(t). Such a procedure is used in the studies of atmospheric gust loading (refs. 21, 22). In applications, however, this approach must be used with caution, since its validity is ensured only when m(t) is

indeed slowly varying compared with $n(t)$. Otherwise, more information about the simultaneous statistical behavior of $m(t)$ and its two derivatives must be known.

Whichever analysis is used, one observes that numerical results for the "peak" distributions depend critically upon particular choices for the density function $p_m(m)$. A particular choice of $p_m(m)$ was made by Hall for modeling of atmospheric radio noise data (ref. 23); for such noise, he argued that a chi-squared distribution for the random variable $b \triangleq 1/m$ should be chosen (Eq. (4)). If so, Eq. (29) has a closed-form expression, namely (using Hall's notation)

$$c^+(y_o) = \frac{1}{\left[1 + \left(\frac{y_o}{\gamma_H}\right)^2\right]^{\frac{\theta-2}{2}}} . \tag{30}$$

where

$$\gamma_H \triangleq \sqrt{M}\,\frac{\sigma_n}{\sigma} \tag{31}$$

$$\theta \triangleq M + 1 . \tag{32}$$

As mentioned earlier numerical choices of values for the parameters (θ, γ_H) in a particular study must await fitting of Eq. (30) to observed data.

Rather than assuming an expression for $p_m(m)$, one might infer its form from observed data. This can be accomplished by rewriting Eq. (29) to obtain

$$c^+(y_o) = \int_0^\infty f(\tau)\, e^{-\frac{y_o^2}{2\sigma_n^2}\tau}\, d\tau \tag{33}$$

where

$$f(\tau) = \tau^{-3/2}\, p_m(\tau^{-1/2}) . \tag{34}$$

Now Eq. (33) can be recognized as a Laplace transform (ref. 24):

$$c^{+}(y_o) = \mathcal{L}\left[f(\tau)\right](s)\Bigg|_{s=\frac{y_o^2}{2\sigma_n^2}} . \tag{35}$$

Therefore, if empirical crest data are fitted with some analytical expression, then $f(\tau)$ is determined by the inverse Laplace transform of the expression[5] for $c^{+}(y_o)$. In principle, this would lead via Eq. (34) to the determination of $p_m(m)$. Such a technique would rely heavily on one's ingenuity in finding an analytical expression $c^{+}(y_o)$, yielding, nevertheless, an appropriate form for $p_m(\cdot)$. In turn, knowledge of $p_m(\cdot)$ could be used to predict other response statistics, e.g., maxima distributions or first-order probability densities which could be used to check the "slowly-varying" aspect of the model empirically. It also should be noted at this point, that for large arguments, Eq. (30) behaves as an inverse power. Thus, if Eq. (30) is to be used as a "peak" exceedance distribution to calculate partial damage moments, one can expect the higher-order moments to diverge.

[5]This approach does not work too well with the Weibull fits to "peak" data reported earlier (ref. 5), since this distribution is seldom transformable in closed form.

IV

ENGINEERING APPROXIMATION: GENERALIZATION OF γ

The preceding sections have dealt with properties of formal analytical solutions for the response crest and maximum statistics. Unfortunately, these results are fairly complicated for nongaussian noise models. Also, for general systems analysis the reliability engineer may not know or be able to determine even such basic information as the system impulse response function $h(\tau)$. The work of Section III-1 reveals that only knowledge of the values of the "peaks" of the system impulse response is absolutely necessary for predicting "peak" statistics in the low impulse limit. However, even this information may be inaccessible at least before the system has been built. One might wonder, then, if some simple, statistically reliable characterization of the linear system under study would not suffice for "engineering" prediction of the response "peak" distributions. In this section two approaches (time-domain and frequency-domain), both based on considerably less system information than such full knowledge about $h(\tau)$, are discussed and applied to predicting response crest statistics. Basically, both methods consist of modeling a complex multimodal system by an appropriate unimodal system chosen in such a way that the level crossings of this "equivalent" system response approximate those of the complex system.

The basis for this system modeling lies in the simple dependence of the unimodal statistics on certain system parameters. As cited earlier, for a 1 DOF system, $\gamma = Q\nu/\omega_o$ completely determines the crest statistics. The system characterization reflected by γ appears only as the ratio Q/ω_o. Thus, in order to model a complex system by a unimodal system in a manner appropriate to predicting level crossings it is first necessary to determine a more general system parameter, analogous to Q/ω_o, for the unimodal system which in turn can be applied to arbitrary systems with an equivalent positive slope zero-crossing rate $\omega_o/2\pi$. These two parameters, then, determine the crest statistics for the "equivalent" unimodal system. It is logical to ask if the crest statistics predicted in Section III-1 for a unimodal system can be used to obtain an approximation to the complex system crest statistics. This possibility will now be explored.

1. TIME-DOMAIN APPROACH

In addition to lending some weight to our hypothesis of the validity of a generalized definition of γ as outlined above, Middleton's work (ref. 8) on Poisson impulsive noise suggests a method of making a meaningful extension of the unimodal parameter Q/ω_o to complex systems. His results predict that, for low impulse frequencies, the nth order response probability densities (and thus the frequency of level crossings via Eq. (7)) depend strongly on the product of the impulse frequency ν and the duration of the system's impulse response, $\bar{\tau}$. This agrees exactly with our unimodal crest results, since Q/ω_o is directly proportional to the time required for the exponential impulse response envelope to decay to a given value and $Q\nu/\omega_o$ determines the system crest statistics. The problem now is to select a convenient, physically meaningful measure of impulse response duration for a more complex system.

The most convenient definition of response duration may well depend on the particular system under study. Initially we restrict our attention to linear multimodal systems having a fairly high number of modes since the empirical results of Section V deal with such systems.

For the statistical analysis of room acoustics acousticians (ref. 25) have found it convenient to describe the room (system) in terms of its reverberation curve (the time history of sound pressure level following cessation of white noise excitation). In particular, the system reverberation time, the 60 dB decay time based on an exponential fit of the initial portion of the reverberation curve, provides a convenient measure of the duration of the impulse response for multimodal systems in general. In order to have application here this "reverberation time", τ_{60dB}, must be converted to an $e^{-1/2}$ decay time τ in conformity with the definition of Q/ω_o as a characteristic decay time for the 1 DOF system. Thus the $e^{-1/2}$ decay time τ is related to τ_{60dB} as follows:

$$\tau = \frac{\tau_{60\ dB}}{6 \log_e 10} . \qquad (36)$$

Using τ as defined above as the generalization of Q/ω_o (i.e., $\gamma = \nu\tau$) yields the crest statistics of the "equivalent"

unimodal system via Section III-1. Methods of approximating the complex system response on the basis of this unimodal system response will be discussed later in the section.

To date no rigorous analytical justification has been found for the choice of the time constant τ obtained from the system's reverberation curve as a meaningful measure of response duration. Empirical verification aside, the reverberation time, being based on the rate of system energy decay in a manner analogous to Q/ω_0 for the 1 DOF system, seems to be intuitively reasonable as a choice for generalizing γ in terms of a time-domain parameter. However, its ultimate justification must lie in empirical verification.

2. FREQUENCY DOMAIN APPROACH

Very often, due to the nature of the system, it is preferable to work with frequency-domain parameters rather than to use the time-domain approach outlined above. In any case, the possibility of a generalized interpretation of γ carries over. The relationship existing between time-domain and frequency-domain concepts merely converts the impulse response duration information into a certain measure of system bandwidth B. As was the case with response duration, τ, system bandwidth is a rather vague term. There exist a variety of definitions for bandwidth in the literature (ref. 26) each defined in a manner useful for the particular problem area involved. For example the "equivalent noise bandwidth" defined by:

$$B_n = \frac{\int_0^{\infty} |H(j\omega)|^2 d\omega}{|H(j\omega)|^2_{max}} , \tag{37}$$

with $H(j\omega)$ the fourier transform of $h(\tau)$, is useful in describing the noise power transmitted by a system under white noise excitation, while the "3 dB bandwidth", defined as the difference between the two frequencies ω_1 and ω_2 at which $|H(j\omega)|^2$ is 3 dB below its peak value, is convenient for describing relatively flat bandpass systems. Most definitions for B that are applicable to the unimodal system yield a bandwidth

$$B = \frac{\omega_o}{Q} \; . \tag{38}$$

For more complex systems, however, the various bandwidth expressions differ, each depending strongly on which particular aspect of the system transfer characteristic is emphasized.

Our immediate objective is to find a definition of bandwidth that is a measure of the inverse of the aforementioned response duration for the highly peaked multimodal response spectra of interest in structural vibrations. Let us now return to the system time-domain response. For the type of system under consideration, response decay is governed by the rate of system energy dissipation. Thus, a bandwidth expression based on such energy considerations is sought. To relate this to the 1 DOF system, consider the fundamental definition of Q (ref. 27):

$$Q = \frac{2\omega_o T}{D} \tag{39}$$

where T is the system time-average kinetic energy and D is its time-average rate of power dissipation. Combining Eq. (39) with Eq. (38) gives the unimodal bandwidth expression in terms of the system energies as:

$$B = \frac{D}{2T} \; . \tag{40}$$

However, the quantities D and T are well defined even for arbitrary systems; hence, Eq. (40) will be used as our general bandwidth definition, being a measure of the rate of decay of system energy.

To obtain useful results from this expression let us specialize our attention to multimodal systems having highly peaked, well defined modes such as the circular metal plate. If the modes are of sufficiently high Q, they can be assumed to be uncoupled, justifying writing the total system kinetic energy as the sum of the individual modal energies T_i (ref. 28). Similarly the total rate of energy dissipation is the sum of the individual rates D_i. Now, for each mode, Eq. (40) can be applied, again assuming high modal Q's, to express D_i in terms of T_i, $D_i = 2B_iT_i$ with B_i the modal 3 dB bandwidths. Thus Eq. (40) for the multimodal system becomes

$$B = \frac{\sum_i B_i T_i}{\sum_i T_i} ; \tag{41}$$

That is, the system bandwidth is just the average of the individual modal 3 dB bandwidths weighted with respect to the modal kinetic energies. A more general interpretation of γ, based on Eq. (40) becomes

$$\gamma = \frac{\nu}{B} = \frac{2\nu T}{D} \tag{42}$$

A certain amount of care must be exercized in applying Eq. (41) to distributed systems in general. In particular, when considering response statistics concern is solely with a particular system input-output characteristics, i.e., the response at a single fixed point, and not with over-all system behavior. From a reliability viewpoint one is concerned only with response t a few selected points of greatest system vulnerability. Hence, each modal energy T_i must not be interpreted as an over-all modal energy summed spatially over the eigenmode shape, but rather as the "localized" modal energy at the response point in question, i.e., the kinetic energy, due to the ith mode, of an infinitesimal mass element at the response point as predicted by the actual system transfer characteristic. If the T_i's are interpreted as modal energies averaged over the plate, Eq. (41) yields an "average system bandwidth" that is independent of the fixed response point and has no significance in so far as the actual system input-output frequency characteristics for the localized response variable under analysis are concerned.

Even with the above two methods for selecting a unimodal system "equivalent" to the given complex system (or more precisely the equivalent system ν) available, the problem of determining a relationship between the actual system crest statistics and those of the unimodal system remains. Since the goal here is to obtain rather simplified engineering approximations, the entire development of this section lacks a rigorous analytical basis, and any justification for the approximations must lie with experiments. Similarly, the

relationship between the statistics of the equivalent unimodal system and the modeled system is based partially on qualitative consideration of the two system impulse responses and ultimately on empirical results. This relationship will be investigated next.

The most convenient relationship between the unimodal and complex multi-modal system statistics, if it yields good approximations, is a direct equality. While a considerable amount of empirical data on various multimodal systems (cf. Section V) indicates that the functional form of the unimodal crest distribution agrees with measured multimodal results, in most cases the two differ by a multiplicative factor of the order of unity. This discrepancy is explained qualitatively by consideration of the impulse responses of the two systems involved. The unimodal response, being sinusoidal in nature, has a zero-crossing preceding each non-zero level crossing. For a multimodal system, on the other hand, in most cases the modes add in such a manner that there may be several extrema of the response (producing level crossings at levels $\eta > 0$) between successive zero-crossings. Thus a unimodal system having the same frequency of level crossings at large response levels as the multimodal system cannot have the correct level crossing rate for small response levels (say $\eta \simeq 0$). Hence while the unimodal system selected as indicated in Sections IV-1 or IV-2 correctly predicts the frequency of level crossings for the modeled system for $\eta > 0.5$, it cannot predict the normalized crest distribution. This is because $\tilde{c}^{+}(\eta)$ is obtained by normalizing the predicted level crossing frequency with respect to the equivalent unimodal resonant frequency (zero-crossing rate) $\omega_o/2\pi$. On the other hand the complex system normalized crest statistics are obtained by normalizing the level crossing frequency with respect to the true system zero-crossing rate, which is slightly smaller than the zero-crossing rate for the unimodal system, $\omega_o/2\pi$, for the reasons stated above.

Thus, the crest predictions differ from the true statistics by a multiplicative factor, the ratio of the two zero-crossing rates which predicts a larger "peak" exceedance probability at higher levels than for the unimodal case. Hence, it appears that in normalizing one should measure or estimate the actual frequency of zero-crossings for the complex system rather than use the zero-crossings corresponding to the equivalent unimodal frequency. Examination of the derivations

in Section III-1 reveals that the level crossing frequency for the unimodal system, $\nu_\gamma^+(\eta)$, is given by

$$\nu_\gamma^+(\eta) = \frac{\omega_o}{2\pi} c_\gamma^+(\eta). \tag{43}$$

Here $c_\gamma^+(\eta)$ is the normalized crest distribution for the unimodal system with impulse density parameter γ. Using Eq. (43) as the predicted level crossing frequency for the complex system and normalizing with respect to the actual (or estimated) complex system zero-crossing rate, $\nu^+(0)$, gives the approximate crest distribution, $c^+(\eta)$, for the complex system:

$$c^+(\eta) = \frac{\omega_o}{2\pi\nu^+(0)} \quad c_\gamma^+(\eta) \tag{44}$$

where γ is obtained by the methods given at the beginning of this section and the equivalent ω_o is yet to be determined.

In a development similar to that of Eqs. (39)-(42) for multimodal systems, Ungar (ref. 28) suggests for this equivalent resonant frequency, ω_o, the center frequency of a band encompassing all modes of the system that give significant contributions to the bandwidth expression Eq. (41). Thus the band is chosen to contain essentially all of the system energy. For the highly peaked spectra studied here this band is well defined; for other systems this might not be the case. A possible alternative expression for ω_o is discussed in Section VI. However, since the center frequency expression above yields good agreement with empirical statistics for multimodal systems it is used for all the results presented in Section V.

The two methods outlined here for selecting the equivalent system γ (and ω_o) are those appropriate to predicting the lightly-damped circular plate response statistics, i.e., methods relying on the highly-peaked, multimodal nature of the system transfer characteristic. For other types of systems, while the exact expressions derived here may not be applicable, the concept of selecting an equivalent γ and ω_o may still be applied, perhaps with some other definition of system bandwidth or response duration more appropriate to the system under consideration.

V

COMPARISONS WITH EMPIRICAL CREST DISTRIBUTIONS

Several methods have been developed in the preceding sections which should enable one to predict with reasonable accuracy the response crest statistics. In this section the application of both the "exact" theories and the "equivalent unimodal system" methods will be discussed. In particular, empirical crest data obtained from impulsive source excitation of a clamped circular plate, along with data obtained from unimodal and bimodal systems, will be compared with the predictions mentioned above.

1. EXPERIMENTAL EQUIPMENT

The analytical impulsive noise model of Eq. (1) is obviously not realizable by a physical noise generator. As reported in previous work (ref. 10) a generator approximating this idealized model has been developed. This generator produces a train of approximately Poisson-distributed, fixed-duration, rectangular pulses whose amplitude distribution $p_a(a)$ is fixed by modulating the pulse train with a random noise having the desired probability density. For all data presented in this section $p_a(a)$ was the normal density. With the pulse width T fixed such that $\frac{1}{T}$ is much greater than the greatest response frequency of the system to be excited, the generator produces a good approximation to the ideal noise model of Eq. (1).

Data obtained using this generator will be presented for three types of systems: unimodal (1 DOF) systems, bimodal systems and the (multimodal) circular plate. The unimodal systems were obtained by synthesizing the equation:

$$\ddot{y} + \frac{\omega_o}{Q} \dot{y} + \omega_o^{\,2} y = f(t) \tag{45}$$

with analog computer modules (ref. 11). The forcing noise is denoted by f(t). This simple system is included only to verify Eqs. (9) and (22). The second type of system was a bimodal system consisting of a pair of elastically coupled oscillators, again synthesized with analog computer modules (ref. 3). Specifically, the equations synthesized were:

$$m_{11}\ddot{y}_1 + c_{11}\dot{y}_1 + k_{11}y_1 + k_{12}y_2 = f(t)$$
$$m_{22}\ddot{y}_2 + c_{22}\dot{y}_2 + k_{22}y_2 + k_{21}y_1 = gf(t) \tag{46}$$

with y_1 the observed response variable and g, m_{jk}, c_{jk} and k_{jk} positive constants. In general these equations produce a system transfer characteristic exhibiting two distinct resonant peaks. Various relative mode strengths were obtained by varying the Q's of the two oscillators. Peak statistics obtained on these two types of systems have been reported previously (refs. 3, 5). The major emphasis here will be on a complex multi-modal system, the clamped circular plate. Details of the plate characteristics are included in the Appendix.

2. "EXACT THEORIES"

a. Poisson Impulsive Noise - Low-ν Case

The analytical methods of Section III-1 can be applied to predicting the response crest distribution of a unimodal system. For these purposes, Eq. (9) and (22) apply. Direct comparisons reveal that the more general result of Eq. (22) agrees in detail with the earlier published results. Figure 2 presents response crest statistics for nine values of the impulse noise density parameter γ; the data can then be interpreted as either experimental results or the predictions of Eq. (9) and Eq. (22); all three being the same for a 1 DOF system (only, of course, for $\gamma \lesssim 0.5$). The gaussian limit, $\gamma \to \infty$, is shown for comparison purposes. One is encouraged, then, to test Eq. (19), from which Eq. (22) is derived, for more complex systems.

The bimodal system provides a convenient system for this test (ref. 3). However, even for this system the form of the impulse response is not readily adapted to the analysis. Thus to evaluate Eq. (19), the maxima and minima of $h(\tau)$ were measured empirically. With the individual modal Q's set at various values in the range $4 \leq Q \leq 20$ it is necessary to retain from ten-to-twenty terms in the summation of Eq. (19). Data for a typical system is compared to the prediction of Eq. (19) in Fig. 3 for four values of ν. Some details of the transfer characteristic for this system are illustrated in Fig. 4. The same general agreement was observed for other

bimodal system transfer characteristics (ref. 3), namely good agreement for the lower impulse frequencies with some deviation for the highest frequency. This discrepancy will be discussed further in Section VI. It should be noted however, even for the highest impulse frequency, that the prediction of Eq. (19) is still far superior to assuming Gaussian statistics, which is currently the only alternative for large impulse frequencies.

It becomes increasingly difficult to apply the result of Eq. (19) to systems more complex than the bimodal case examined above. Consider, for example, the response crest statistics of a clamped circular plate forced with impulsive noise. Here the impulse response for such a system is extremely complicated, and one has no detailed knowledge of $h(\tau)$. In this case, Eq. (19) would be inapplicable and other methods for examining the response crest statistics must be found.

b. Generalized-t Impulsive Noise Model

The crest statistics of a noise process characterized by Hall's model (ref. 7) (Section III-2) can be readily computed. In spite of the shortcomings of his model (primarily, the inability to determine a priori the parameters γ_H and θ) it may nevertheless serve as a starting point for determining an analytical form for the response crest statistics of a particular multimodal system.

To this end, then, γ_H and θ can be determined by fitting Eq. (30) to some typical response data. This task is better accomplished after first noting that the ultimate application is the calculation of the system damage moments. As a result it will probably be most important to fit the data for the larger response levels where the contribution to these moments is most important. Figure 5 indicates the results of such an operation for three typical cases of multimodal (i.e., clamped circular plate) response crest statistics. The various system transfer characteristics of these three typical cases are shown in Fig. 6.

The agreement between Eq. (30) and multimodal experimental data is quite remarkable. Although some discrepancies do exist at the lower response levels, they make little difference insofar as damage calculations are concerned. And, in fact, since the fitted curve seems to lie above experimental

data for these lower response levels, the analytical moments should provide a conservative estimate of the damage. By and large such a "fitting" procedure parallels earlier attempts to describe response data with Weibull distributions (ref. 5). However, unlike the case of the Weibull distribution, the evaluation of γ_H and θ here will critically delimit the convergence of even lower-order damage moments which "physically" are known to be finite. Thus, more appropriate methods of characterizing the response crest statistics of multimodal systems are sought.

3. EQUIVALENT UNIMODAL APPROXIMATIONS

Even though the exact results of Section III yield very good predictions of general system peak statistics, in many systems they do not apply. Even for a bimodal system their applicability generally relies on an empirical determination of $h(\tau)$. For systems such as the high-Q circular plate, empirical measurements of $h(\tau)$ are impractical. Thus, in this section, the approximate methods derived in Section IV will be applied to the circular plate and to the bimodal systems of Section V-1.

For this application, it is necessary to determine either the system reverberation time τ or the system bandwidth of Eq. (41) in order to apply the results of Section IV. For the circular plate the reverberation time based on the initial decay slope was measured using standard techniques (ref. 29) with a Bruel and Kjaer level recorder and noise generator. For reasons to be discussed in Section VI this equipment was not suitable for measurements of the reverberation time of the bimodal system. To date no acceptable alternative method has been found for measuring decay time of a bimodal system. Hence, all predictions will be based on the system bandwidth, Eq. (41). For the circular plate, results will be presented based both on the reverberation time and the system bandwidth.

Before presenting the data let us consider the bandwidth prediction. In evaluating the bandwidth of the circular plate from Eq. (41) it is necessary to determine the modal kinetic energies at the response point. It is obvious from a typical system transfer characteristic (cf. Fig. 6) that the vibrational energy of a particular mode of the plate is concentrated in a narrow frequency band about that modal resonance frequency and thus has negligible response at any

other modal resonant frequency. Thus, to the degree of approximation involved in the method of Section IV each particular mode will be assumed to be a high Q resonant peak similar to the 1 DOF system. From this assumption the modal kinetic energies become:

$$T_i = \frac{1}{4} m P_i B_i \omega_i^2 \tag{47}$$

where P_i is the absolute response mean square at the resonant frequency ω_i and m is the surface mass density at the response point. Thus, the bandwidth of the system becomes:

$$B = \frac{\sum_i P_i B_i^2 \omega_i^2}{\sum_j P_j B_j \omega_j^2} . \tag{48}$$

From this expression one gains some insight into the relative significance of the various response modes. In particular, for a typical system, with B_i roughly proportional to ω_i, the modal bandwidths are weighted approximately by the third power of modal resonant frequency. Thus a high frequency mode whose response is an order of magnitude or more below the fundamental mode will have a much more significant contribution to system bandwidth than will the fundamental. Indeed, for the system of Fig. 6-b the contribution of the fundamental mode to Eq. (48) is insignificant compared to those of the sixth and seventh modes whose responses are 20 dB lower.

Several multimodal transfer characteristics for the circular plate were studied, obtained by using various pickup probe positions and various amounts of vibration damping tape. Selected crest data so obtained (Propagation Research Laboratory impulsive crest runs numbered 117-119) are presented in Figs. 7-9. The three figures represent the system illustrated in Fig. 6-b excited at three different values of average impulse frequency ν = 15, 5, 1. Also shown in these figures are the crest statistics predicted by the reverberation time and bandwidth methods discussed above. Generally, the agreement between the predictions and the data is fairly good in the range $\eta \gtrsim 0.5$, probably within a factor of two to three, for the nongaussian situations

represented ($\gamma \leq 0.5$). Also, the agreement obtained via the two methods is roughly the same; one difference is that the reverberation time predictions favor more Gaussian behavior[6] while the bandwidth predictions are exemplary of slightly more nongaussian behavior. Thus, the former seemingly provides a lower, while the latter an upper bound for reliability predictions. Both methods yield good approximate crest statistics; the reliability engineer seeking conservative estimates of system lifetime would find the bandwidth prediction more applicable than the reverberation time prediction.

The multimodal bandwidth expression Eq. (48) also will be used to predict the bimodal system crest statistics although more drastic approximations are involved in this case. In general, the Q's of the bimodal system response peaks are not sufficiently large to validate the assumptions leading to Eq. (48). Moreover, the response peak heights and bandwidths measured from the system transfer characteristic are not the true modal parameters. In general the approximations involved tend to weight the bandwidth of the high frequency mode too strongly in Eq. (41). Since the bandwidth of this mode is generally larger than the other, Eq. (48) tends to predict a larger system bandwidth than expected. This, then, corresponds to a conservative prediction of the system crest statistics insofar as damage calculations are concerned.

The approximate crest predictions of Section IV using the bimodal bandwidth of Eq. (48) and ω_o as defined in Section IV are compared with bimodal crest data in Fig. 10. It should be noted that for all but the highest impulse frequency the deviation between the data and the approximate prediction is less than a factor of two. Also the predictions of Section IV are slightly too nongaussian using the bandwidth expression, Eq. (48), as might be expected.

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[6] I.e., the "reverberation time" tends to be too large (decay constant too small).

VI

LIMITATION OF THEORETICAL PREDICTION OF CRESTS

Little need be said here about the agreement of the "exact" theories of Section III-1 with experiment for a unimodal system excited by impulsive noise. By in large, earlier work (ref. 3) has shown Eq. (22) to be valid for sufficiently small values of the impulse frequency ν. It is important to point out, however, that the theory in Section III-1 summarized by Eq. (19) (specifically for a unimodal system Eq. (22)) has been found to agree with Eq. (9) for values of the impulse-noise-density parameter γ. This success encouraged a test of the more general "exact" theory of Eq. (19) by applying it to a more complex system and also encouraged attempts to extend an interpretation of γ to complex multi-modal systems.

As an initial step toward reliability predictions for complex systems the bimodal case was considered (ref. 5). Again, it has been shown (ref. 3) that Eq. (19) is a most acceptable predictor for the response peak statistics. Although it is most accurate in the low impulse-frequency range, Eq. (19) also can provide estimates of the response crest statistics for larger values of ν. The real difficulty lies in an inability to explicitly state how accurate Eq. (19) is for a given value of ν in a general way. For the unimodal case the parameter γ incorporates both the noise-source parameter ν and the system decay time (reciprocal bandwidth) parameter Q/ω_o into the prediction. But, for a multimodal system "bandwidth" is not a clear-cut, well-defined parameter. It appears, then, that until such a parameter becomes well-defined it may not be possible to describe the general case in a straightforward manner.

One should not, however, at this point dismiss the results of Eq. (19) as being purely academic. In the highly nongaussian situations it serves for the present as an accurate predictor for the response peak statistics of a multi-modal system. Knowledge of the values of the maxima and minima of $h(\tau)$ is sufficient for the application of the technique. So interpreted, one is not required to know every detail of $h(\tau)$. This technique will therefore be most useful, when one is confronted with estimating the time to failure of a particular system.

Although application of the result Eq. (19) is straightforward for low impulse frequencies when the summation rapidly converges, empirical data indicate that Eq. (19) can be applied for impulse frequencies slightly larger than those assumed in the derivation. For such applications, however, in order to avoid summing over extrema occurring after the beginning of the next impulse response it is necessary to truncate the summation to include only those extrema occurring before $t = \frac{1}{\vartheta}$. This truncation procedure (which eliminates various low-amplitude peaks) is included in the results of Eq. (22).

In an effort to overcome the need for detailed knowledge of $h(\tau)$, Hall's noise model was considered in an attempt to characterize system response. For lack of justification of the use of any other modulating density function $p_m(\cdot)$, Hall's generalized-t case was examined. This led to a convenient closed form expression for the crest statistics, Eq. (30). Unfortunately, though, the parameters γ_H and θ can only be determined by fitting Eq. (30) to actual response crest data. And then, it is possible that infinite damage predictions may result. Thus again one is restrained, and the application of Hall's model must proceed on a "case-by-case" approach. Table I below illustrates one such attempt. The data ranges from a highly nongaussian case (run 107) to a moderately Gaussian one (run 37). The resultant

Table I Values of Hall's Parameters Determined by Fit to Experimental Crest Data

run	θ	γ_H
37	10.0	2.9
118	6.2	1.65
107	3.8	1.12

calculation of θ and γ_H suggests that both parameters decrease as more nongaussian processes are encountered. Of the two, variations in the parameter γ_H are perhaps of lesser importance since they basically represent a change of scale for the response level. Of greater significance are the changes in θ. The behavior of Eq. (30) for large y_o is inversely proportional to y_o raised to the $(\theta-2)$ power. If Eq. (30) is to be used for the calculation of damage moments

(see Section VII), then there appear to be some damage moments which will no longer be finite. In fact, if the highest-order finite damage moment is of order n, then necessarily

$$n < \lceil \theta - 2 \rceil \tag{49}$$

where $\lceil \cdot \rceil$ is the greatest integer operation. Thus, if one assumes the fit of Hall's analytical model to be correct for the highly nongaussian case of run 107 a second-order damage moment does not exist. And, even for the "more Gaussian" run 37, the damage moments of order less than eight are the only ones that are finite!

This, of course, is a physically unrealistic situation. The point that should be made is that although Eq. (30) can be adjusted to fit typical response data quite nicely, it is not indicative of the true system response at the larger response levels, say greater than ten times root-mean-square. One could apply some truncation procedure to Eq. (30); such an operation would need to be systemized, however, after more careful study of each independent case.

It appears, then, that insofar as damage calculations are concerned, Hall's "generalized-t" model is inadequate to describe the reliability of linear systems forced by impulsive excitation. Future work may suggest some other modulating density function $p_m(\cdot)$ which will lead to more acceptable analytic response crest predictions. Until this time, one should examine other alternate approaches which may cast light on the behavior of the response statistics.

Although the "equivalent unimodal system" methods of Section IV provide only an approximation to the response statistics, they appear to offer greater utility for the reliability engineer than do the more exact methods discussed above. The concept of an equivalent γ parameter makes possible broad generalizations concerning the response statistics. In a qualitative sense the statistics become more nongaussian as γ decreases, corresponding to a decrease in the impulsive rate or equivalently to a decrease in system response duration or an increase in system bandwidth. In addition these methods yield a fair (order of magnitude or better) quantitative approximation for the statistics on the basis of very limited system information, i.e., γ/ν and ω_o. Thus, such order of magnitude predictions are

thought to be invaluable in designing complex systems whose exact response characteristics cannot be known a priori. This conjecture requires further study.

There are, of course, some limitations to the applicability of our methods in the form presented here. While the concept of an equivalent system γ and the resulting qualitative statements are generally useful, for many situations the systems reverberation time or the bandwidth defined by Eq. (41) are not clear. And, as was mentioned in Section V, the standard methods of measuring reverberation time may fail even for a bimodal system. In this study the rapid decay of the impulse response of the bimodal system exceeded the writing-speed limits of the mechanical chart recorders. Indeed for low $Q(Q < 10)$ systems in general, the utility of a reverberation characteristic is in doubt since the decay curve is not well-defined. Rather than a smooth decay the curve exhibits a series of rapid falls and "plateaus" making the determination of a single rate of decay rather arbitrary. For such systems some other meaningful measure of response duration must be developed.

Like reverberation time, the bandwidth concepts investigated here are limited in range of applicability. For example, Eq. (48), is restricted to systems having highly peaked transfer characteristics exhibiting well-defined, relatively independent modes. For other types of systems the more general expression of Eq. (40) would be more appropriate since D and T are defined for all systems. Indeed for some systems an entirely different bandwidth definition might have more utility especially if system energies are not readily available. In selecting a definition of bandwidth, however, one should keep in mind that basically it must be a measure of the inverse of characteristic system response time.

There also are some difficulties with the particular choice of ω_0, the center frequency of a band encompassing all significant modes of the system, required to normalize crest statistics. For more general systems, the choice of such a band of frequencies may not be straightforward. In such cases perhaps a more satisfactory choice for $\omega_0/2\pi$ might be the zero-crossing rate for the complex system excited by Gaussian white noise (e.g., Poisson impulsive noise in the limit $\nu \rightarrow \infty$). Thus the resonant frequency becomes (ref. 30):

$$\omega_o \equiv 2\pi \lim_{\nu \to \infty} \nu^+(0) = \left[\frac{\int_0^\infty |H(j\omega)|^2 \omega^2 d\omega}{\int_0^\infty |H(j\omega)|^2 d\omega} \right]^{1/2} \tag{50}$$

with $H(j\omega)$ the complex system transfer characteristic. Again, the utility of definitions like Eq. (50) must rely ultimately on empirical verification. It is noted for the multimodal systems examined in this study that Eq. (50) is roughly equivalent to the "center frequency" defined in Section IV.

Prediction of the appropriate exact positive-slope zero-crossing rate, $\nu^+(0)$, is another problem that arises when one normalizes the crest distribution. For overload predictions this is unnecessary, but for wearout prediction the normalized crest distribution is treated as the "peak" distribution. One surmises that probably the damage over-estimate inherent in the use of the normalized crest distribution (but the "peak" occurrence frequency for the rises) will persist even after the extra factor $\omega_o/2\pi\nu(0)^+$ is dropped. At any rate, more work remains to be done here, as in the other matters connected with the "equivalent" unimodal model.

The entire development of Section IV hinges on empirical justification. Thus while the procedures outlined for selecting the "equivalent" unimodal system yield good agreement with multimodal data, they do not preclude the existence of other, more accurate methods but merely indicate the utility of the concept of an "equivalent" unimodal system.

VII

IMPLICATIONS FOR RELIABILITY PREDICTION

The bewildering diversity of failure modes encountered in engineering practice apparently can be classified under two headings (refs. 4, 5): "overload" or "wearout" failures in terms of appropriate system response variables. Thus, in quite a variety of cases, it is possible to predict system failure rates from mathematical models for the two generic types of failure mechanisms, using knowledge of certain response statistics, through the concept of a wear-dependent failure rate.

So, for overload failure occurring at response level $y_c(\Delta)$, which in general may be a function of total wear Δ (at the weakest point), the wear-dependent failure rate can be approximated (refs. 4, 5) as

$$h_o(t|\Delta) = N^+(y_c(\Delta);t) \ , \tag{51}$$

where $N^+(y_c(\Delta);t)$ is the average frequency of response level crossings at the weakest point in the system. (The wear-dependent failure rate, $h(t|\Delta)$, can be defined as the probability density of failure of the system at time t, given perfect operation to date, and given total wear Δ at said time.) Similarly the wear-dependent failure rate for wearout failure can be expressed as (refs. 4, 5)

$$h_w(t|\Delta) = \nu_p \bar{F}(\Delta^* - \Delta) \ , \tag{52}$$

where ν_p is the average frequency of response peaks, while $\bar{F}(x)$ is the exceedance probability of partial wear (per response "cycle"), obtained from response "peak" statistics, and Δ^* is the wearout level (usually unity). Generally, the Palmgren-Miner linear wear accumulation law is used, and the partial wear, x, is expressed as a power of the peak value S, i.e.,

$$x = A_o S^{\alpha} \tag{53}$$

for some $A_o > 0$ and $\alpha > 1$, both constants fixed by the material (ref. 1). Alternately, the wear accumulation process can be "linearized" by judicious choice (ref. 1) of expression

for the analytical relation between "peak" values and partial wear.

To get the overload and wearout failure rates proper, one must average $h_o(t|\Delta)$ and $h_w(t|\Delta)$, using the total wear probability density for the operational system in question. Ordinarily, one makes use of a renewal-process-wear model (refs. 32, 33), which leads to an asymptotic normal wear distribution for a system chosen at random, and truncated normal wear (refs. 4, 5) (truncated at Δ^*) in an operational system having a definite wearout level.[7] The failure rates may be added to obtain a conservative estimate of system reliability. Note that, all these numerical estimates using wear-dependent failure rates--even apart from the question of finding more exact total wear distribution--call for the use of a digital computer in reliability analysis (ref. 5).

To obviate the need for extensive digital computations, one tries using the "equivalent single mode" approach, hoping to obtain "standard" curves of reliability (wearout and overload separately, say) for the impulsive noise environment in question solely on the basis of unimodal statistics. In rough wearout reliability computations, of course, a wearout distribution, derived from the truncated normal wear distribution which needs only knowledge of the mean and mean square wear at any time (in a system chosen at random) may be sufficient.[7] But for more refined estimates, without overescalating the computations, one would utilize "universal curves" or tables for reliability estimates, for the "equivalent" unimodal system undergoing a statistically identical loading. Such data, presumably, can be readily obtained empirically, and thereafter collected systematically.

Unfortunately, the situation is not so cut and dried: the anticipated equivalence is not that "equivalent" judging by the mean and mean square partial wear values such as shown in Table II (calculated for $\sigma_y = 1$, $A_o = 1$, $\alpha = 4$). Here the equivalent γ and ω_o values are tabulated for the bimodal and multimodal cases discussed earlier (Figs. 3,7-9). One notes

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[7] It has been found that the averaging process introduces a further conservative correction (refs. 4, 5) in system reliability compared with using the normal wear exceedance probability evaluated at Δ^* directly as a wearout reliability.

Table II Mean and Mean Square Partial Wear

γ^*	γ_d	$\dfrac{\omega_o}{2\pi\nu^+(0)}$	$\langle\eta^4\rangle$	$\langle\eta^8\rangle$	θ	β	Run no. (if multi-modal)
0.01	0.01	1.61	28,890	1.046×10^{15}	0.154	0.367	119
0.014	0.014	2.69	4,215	8.696×10^{10}	0.832	0.550	
0.05	0.05	1.9	265.3	2.193×10^{7}	1.13	0.773	
0.047	0.075	1.40	40.42	4.283×10^{7}	0.637	0.689	118
0.136	0.136	1.05	58.08	2.939×10^{5}	1.16	0.956	
0.14	0.17	1.46	19.46	5.22×10^{4}	0.893	0.909	117
0.458	0.458	0.57	12.46	9.465×10^{3}	0.887	1.03	
∞	∞	1.00	1.88	131.7	1.414	2.00	

also a fluctuation in the normalization constant $\omega_o/2\pi\nu^+(0)$ for the crest distribution, indicating the aforementioned empirical versus equivalent unimodal zero-crossing rate discrepancy (the "theoretical" value being $\omega_o/2\pi$). The γ_d column indicates the γ yielding "optimum" fit, in the experimentalists' opinion, while γ^* is the value obtained using the "bandwidth" procedure discussed in Section IV. The θ and β columns reveal that a Weibull fit (ref. 5) was needed to estimate the contribution of the "tail" (outside experimental data range) of the system crest statistics to partial wear. Thus, while in the main the "trends" seem correct, there are various fluctuations from a "neat" two-parameter theory,[8] warning that the single mode equivalence has limits.

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[8] Strictly speaking, it is not γ and ω_o, but γ and $\omega_o/2\pi\nu_o^+(0)$ that are needed in wearout (as compared with overload) predictions. Moreover, one needs an estimate for the average frequency of peaks ν_p, for which purpose $\nu^+(0)$ is generally too small; the average frequency of rises will provide a conservative estimate for ν_p. Thus, the time scales in any "standard" reliability curves must be properly interpreted.

To underscore the importance of the "tail", the following table gives the percent contribution of the tail to the multimodal moments cited above; under identical conditions:

Table III Percent Contribution of the Extrapolated Region to Partial Wear Moments

γ^*	Run	mean	mean square
0.14	117	7.7%	56.9%
0.047	118	46.8	97.3
0.01	119	99.76	100.0

Thus, the data regions outside $0 < \eta < 10$ increase in their importance as the response statistics become more "impulsive" in their deviation from the Gaussian.

This is further emphasized by comparing the results of a graphi:al extrapolation moment estimation technique (utilizing a planimeter) with the Weibull fit (corrected for deviation inside the empirical data range) for the aforementioned cases (of Table II); as shown in Table IV:

Table IV Wear Moments: "Adjusted Weibull" versus "Graphical Extrapolation"

γ^*	$\frac{\omega_o}{2\pi\nu^+(0)}$	$\langle\eta^4\rangle$		$\langle\eta^8\rangle$	
		AW	GE	AW	GE
0.01	1.61	28,890	102.0	1.046×10^{15}	1.01×10^{5}
0.014	2.69	4,215	1040	8.696×10^{10}	7.7×10^{6}
0.05	1.9	265.3	224	2.193×10^{7}	1.8×10^{6}
0.047	1.40	40.42	55.7	4.283×10^{7}	5.29×10^{4}
0.136	1.05	58.08	49	2.939×10^{5}	4.2×10^{4}
0.14	1.46	19.46	36.8	5.22×10^{4}	2.24×10^{4}
0.458	1.00	12.42	12.7	9.465×10^{3}	7.18×10^{3}

The results of the last three tables speak for themselves: Clearly, it would be presumptuous to say too much about unimodal equivalence and its validity for wearout (and overload) prediction, until the "tails" of the crest distributions have been more thoroughly studied.[9]

Various other aspects of reliability prediction might be further explored in this same vein, for impulsively-driven linear systems, but the above estimates of the margin of extrapolation should drive home the tentativeness of these considerations: much more work remains to be done in this area. However, despite its limitations, the equivalent unimodal approach seems to be quite promising from an engineering standpoint, and until better models for impulsive noise are found, it can be used for obtaining engineering bounds on system reliability, once a library of unimodal reliabilities has been compiled. (Of course, some time scaling will always be necessary, owing to the ratio $\nu_p/\nu^+(0)$ not being unity, prior to the multiplication of the wearout and overload reliabilities to yield system reliability.) It is hoped that future investigations will further explore the leads provided in the present study, leading to more complete understanding of the perplexing behavior of nongaussian response processes, and their influence upon system reliability.

[9] A two-year program for more detailed study of unimodal equivalence has been outlined by the staff of the Propagation Research Laboratory, Department of Electrical Engineering, University of Minnesota, in its "Proposal for Research in System Statistical Response and Reliability," dated April 30, 1967.

APPENDIX

CLAMPED CIRCULAR PLATE

There are many possible systems which could be used for the purpose of analysis of a multimodal system forced by impulse noise. The clamped circular plate was selected as such a system for study for several reasons. First, it could be fabricated with relative ease at the machine shop facilities of the Electrical Engineering Department, University of Minnesota. Second, the transfer characteristics of a clamped circular plate bear semblance to the actual systems with which this work might be ultimately applied, say, for example, for fatigue estimates of an airframe panel. Third, the clamped circular plate lends itself readily to analysis (ref. 31). This feature expedites the design of the plate and allows one to place the plate resonant frequencies within a workable range.

Figure 11 illustrates the circular plate which was constructed along with its peripheral steel support ring, forcing assembly, and sensing transducer. High-leaded brass was selected as the plate material after an initial attempt to fabricate the plate from aluminum failed. The fundamental frequency was 336 Hz as compared with the theoretical resonance of 285 Hz. The agreement between predicted resonant frequency and that actually observed improved markedly for the higher modes.

The driving transducer was from a Jensen speaker assembly that had been used in previous unimodal studies (ref. 34). Although this forcing transducer could be situated at four different positions, only the one located at a distance of two thirds the plate radius from the center was considered in the work reported herein. This particular configuration seemed to excite the largest number of modes and therefore constituted a "good" multimodal case.

The sensing transducer is a constant-charge electrostatic device which produces an output voltage proportional to the displacement of the plate. A Bruel and Kjaer microphone amplifier and cathode follower provides the constant-charge circuitry (ref. 35). The pickup probe (see Fig. 12), then, can be mounted directly on the end of the B and K cathode follower unit. A support assembly allows the positioning

of the probe along the center line and rigidly holds it close to the plate surface, say approximately 0.005 inches away. The electrical insulation isolating the support assembly from the plate proper allows one to determine via a continuity check whether the probe is positioned sufficiently far from the plate so as to disallow physical contact between the probe and the plate. This electrostatic-probe detection system was found to be more than sufficient to detect accurately the forced vibrations of the plate.

The particular alloy of leaded-brass used exhibits very little internal damping. This fact necessitates the application of damping tape to the plates surface whenever lower modal quality factors are desired. The forcing transducer can also provide some electromagnetic damping, but this effect is negligible when the plate is forced at the two-thirds-radius position.

Table V summarizes the dimensions of the clamped circular plate as well as listing the material specifications for the brass used. In addition, Table VI indicates the modes which were significant in the system transfer characteristic (see also Fig. 6-a-b-c). The index m designates the number of modal diameters while the number of circular modes is denoted by n. Also shown in Table VI are the theoretical eigenfrequencies obtained by utilizing the information in Table V.

Table V Physical Constants for Experimental Plate

Material: high-leaded brass alloy #260

Density: $\rho = 8.55$ gm/cm^3

Young's Modulus: $Y_o = 11.0 \times 10^{11}$ dynes/cm^2

Radius: 9.50 inches

Thickness: 0.092 inches

Table VI Resonant Frequencies of Experimental Plate

mode (m,n)	0,1	1,1	3,1	4,1	0,3	6,1
theo. res freq., Hz	285	595	1420	1920	2440	3210
actual res. freq., Hz	337	634	1410	1890	2440	3040

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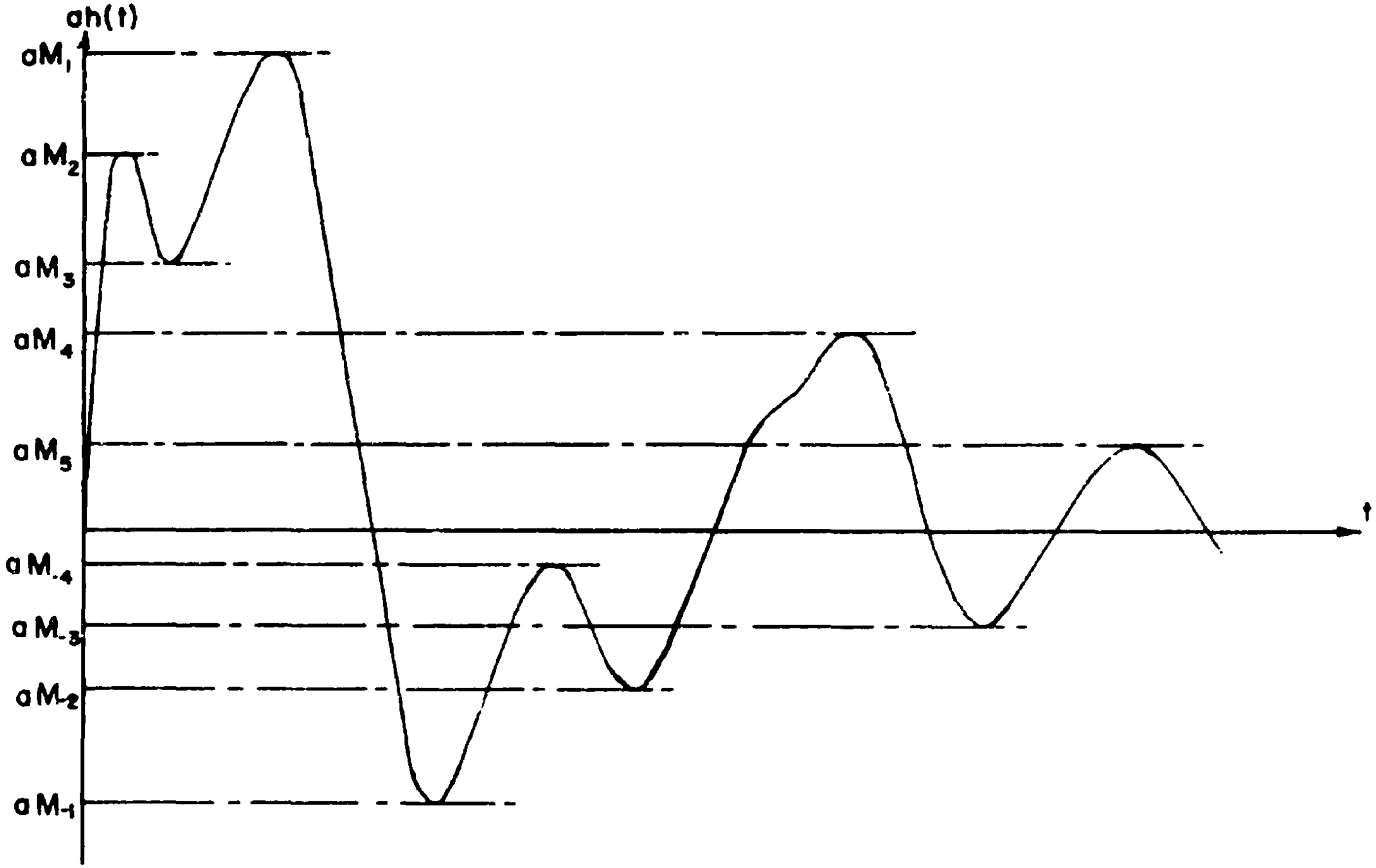
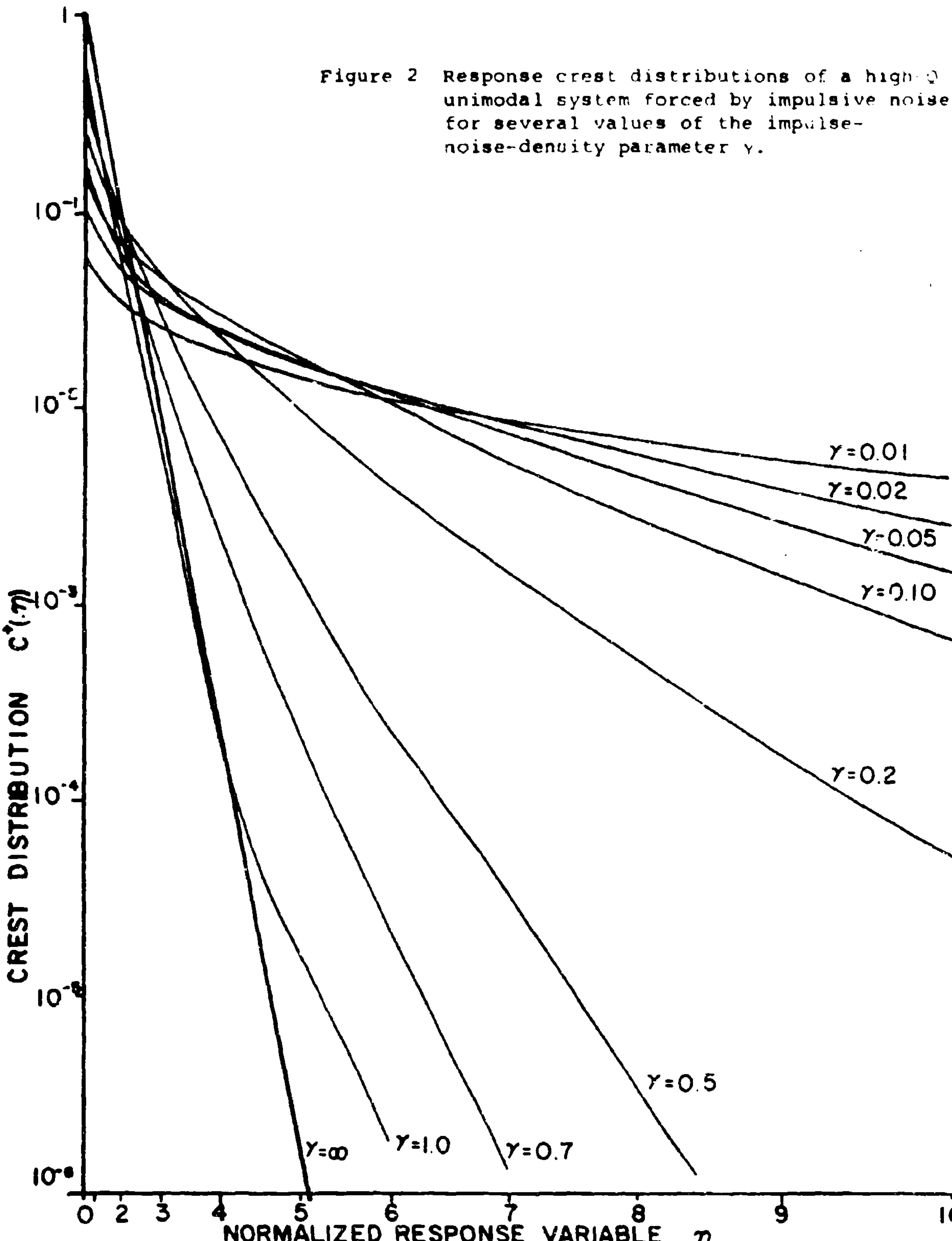


Figure 1 Response of a general time-invariant linear system to an impulse of strength "a" with labeling procedure for extrema.

Figure 2 Response crest distributions of a high-Q unimodal system forced by impulsive noise for several values of the impulse-noise-density parameter γ.



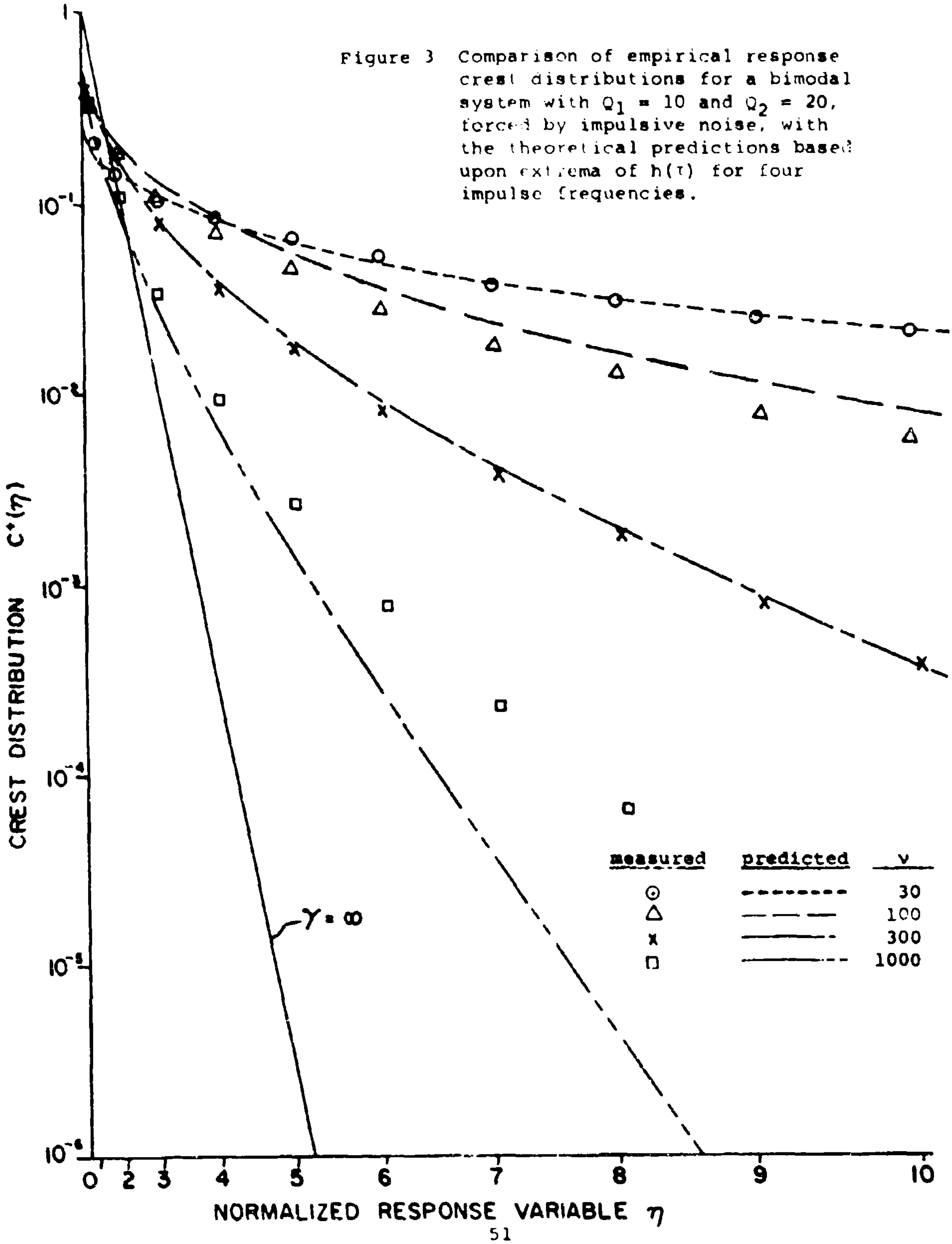


Figure 3 Comparison of empirical response crest distributions for a bimodal system with $Q_1 = 10$ and $Q_2 = 20$, forced by impulsive noise, with the theoretical predictions based upon extrema of $h(\tau)$ for four impulse frequencies.

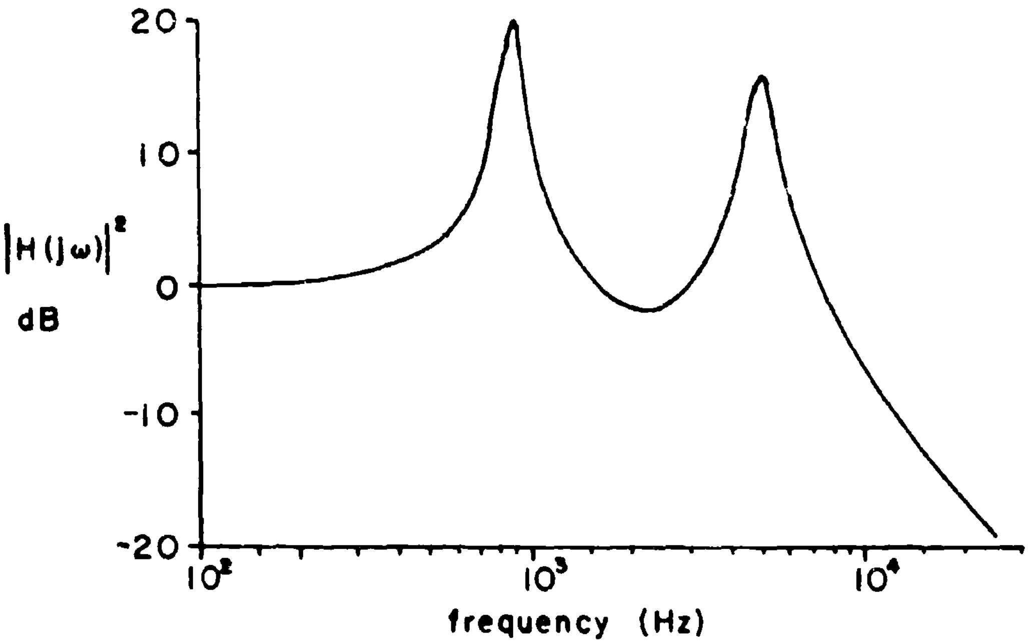


Figure 4 Transfer characteristics of a bimodal system with $Q_1 = 10$, $Q_2 = 20$, normalized to 0 dB at zero frequency.

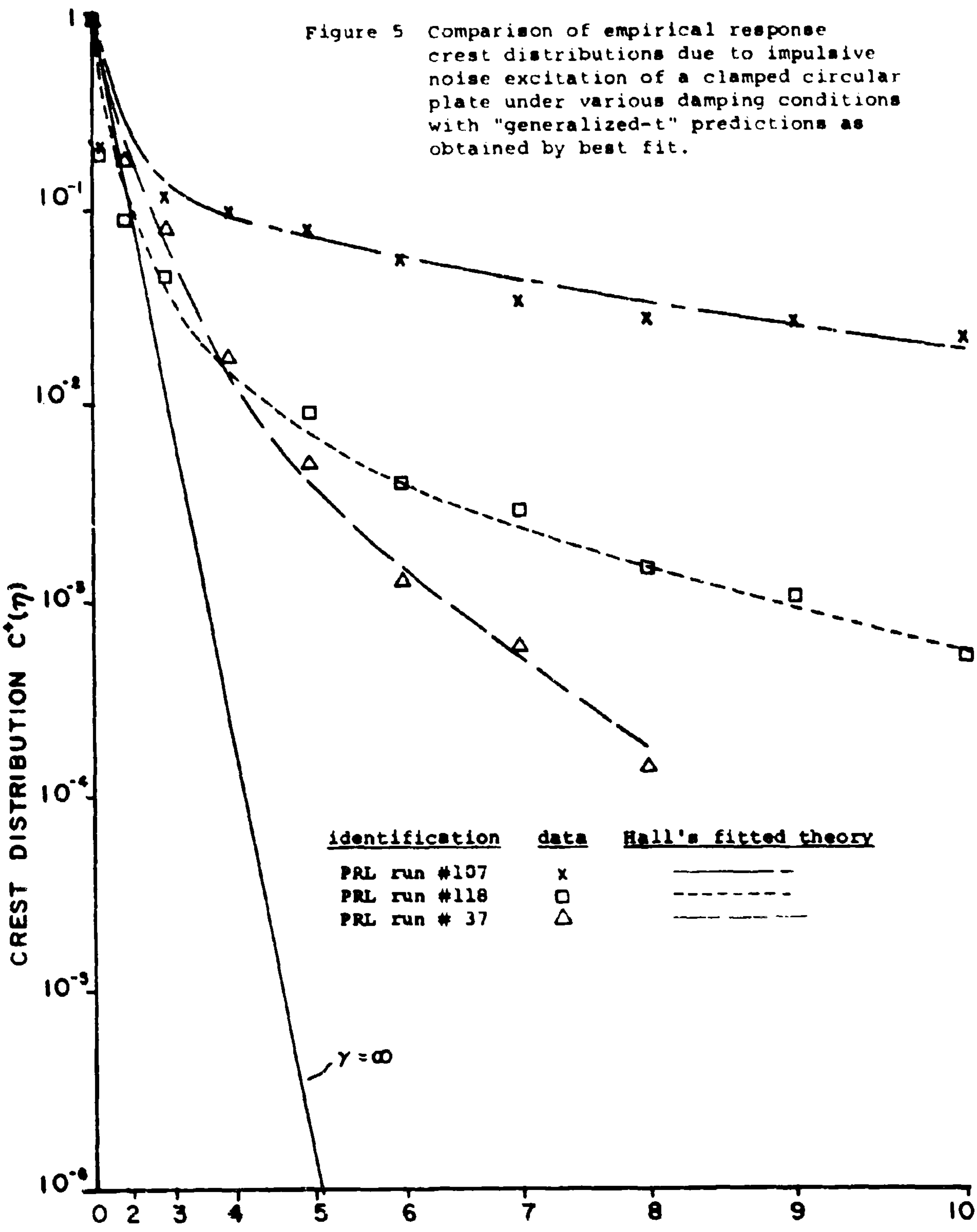
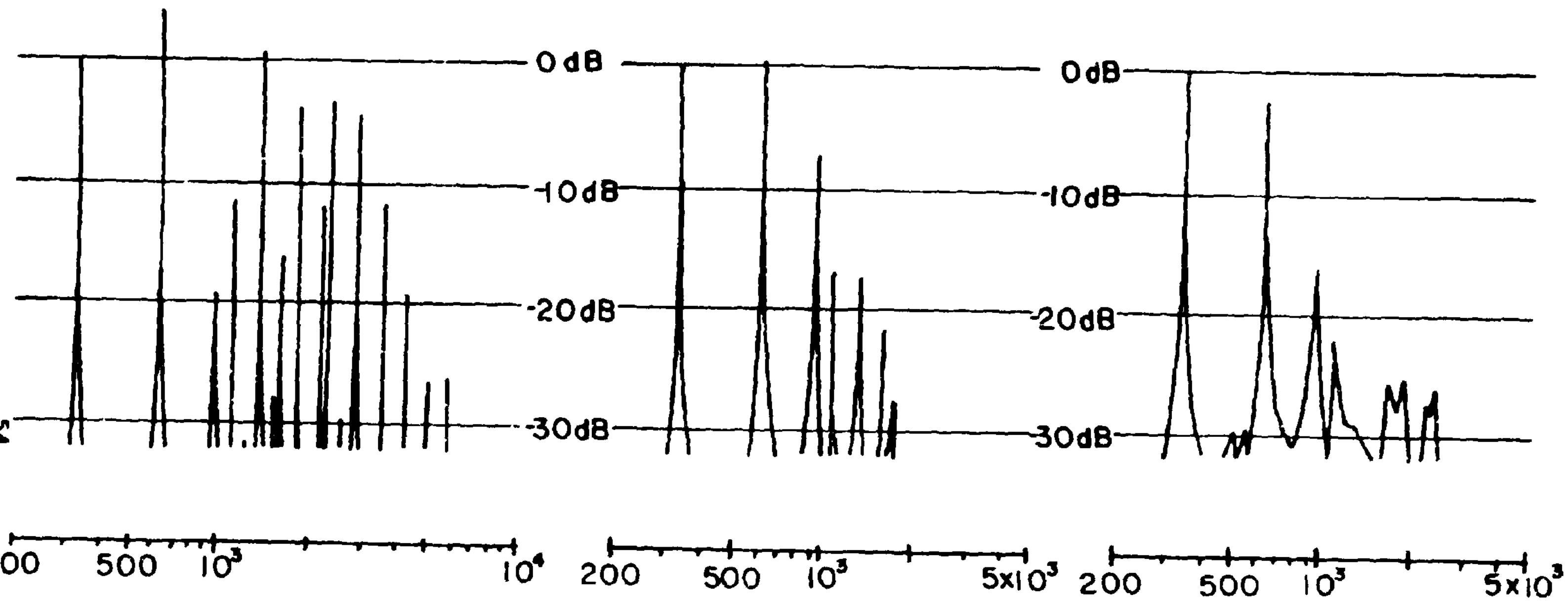


Figure 5 Comparison of empirical response crest distributions due to impulsive noise excitation of a clamped circular plate under various damping conditions with "generalized-t" predictions as obtained by best fit.



a) "Many modes"; undamped. Sensing probe at r = 4.031 inches

b) "Few modes"; slightly damped. Sensing probe at r = 4.031 inches

c) "Few modes"; heavily damped. Sensing probe at r = 4.031 inches

Figure 6 Transfer characteristics of a clamped circular plate under three damping conditions when forced at the two-thirds radial position with sensing probe at the near-edge location 180° from the forcing radius.

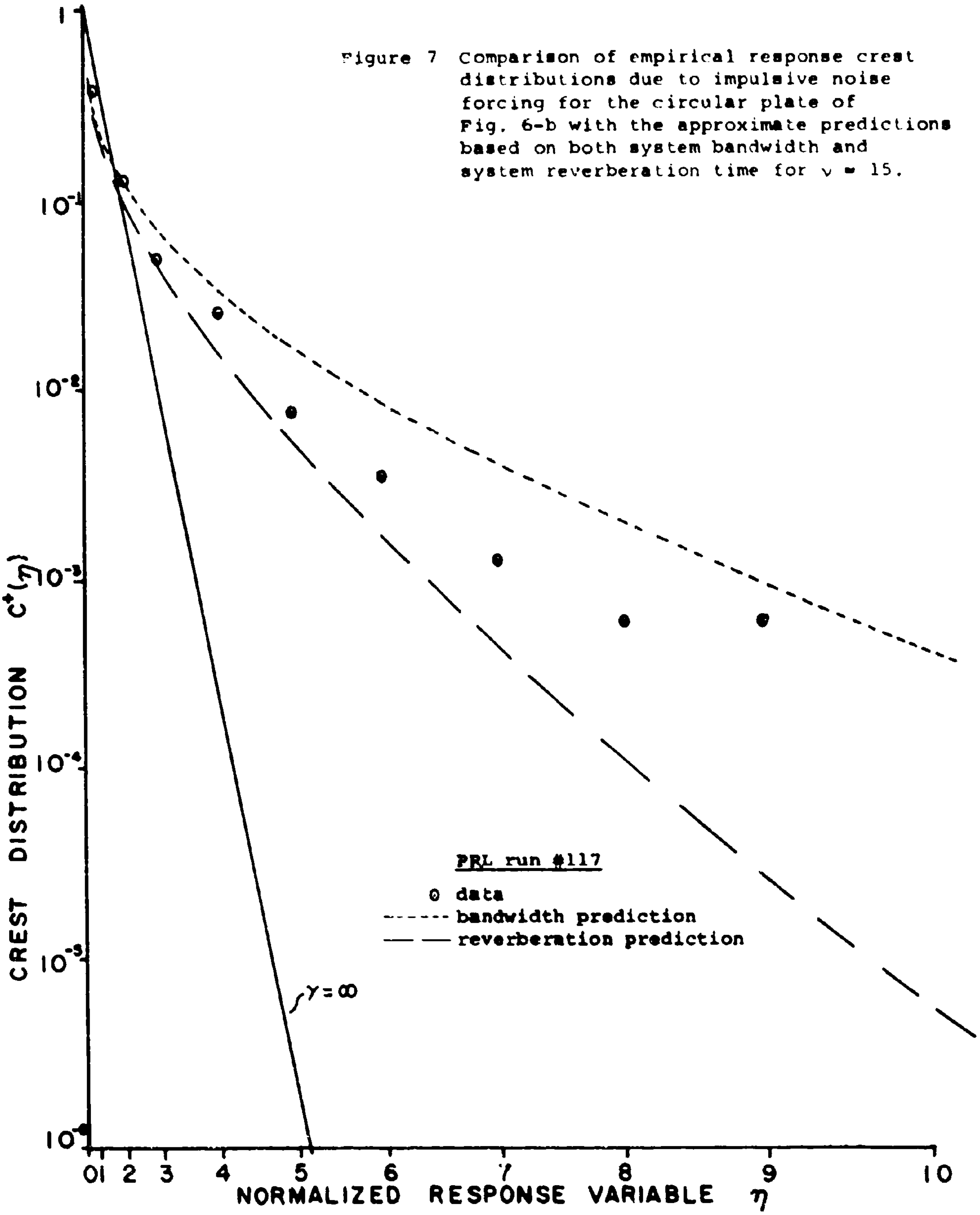
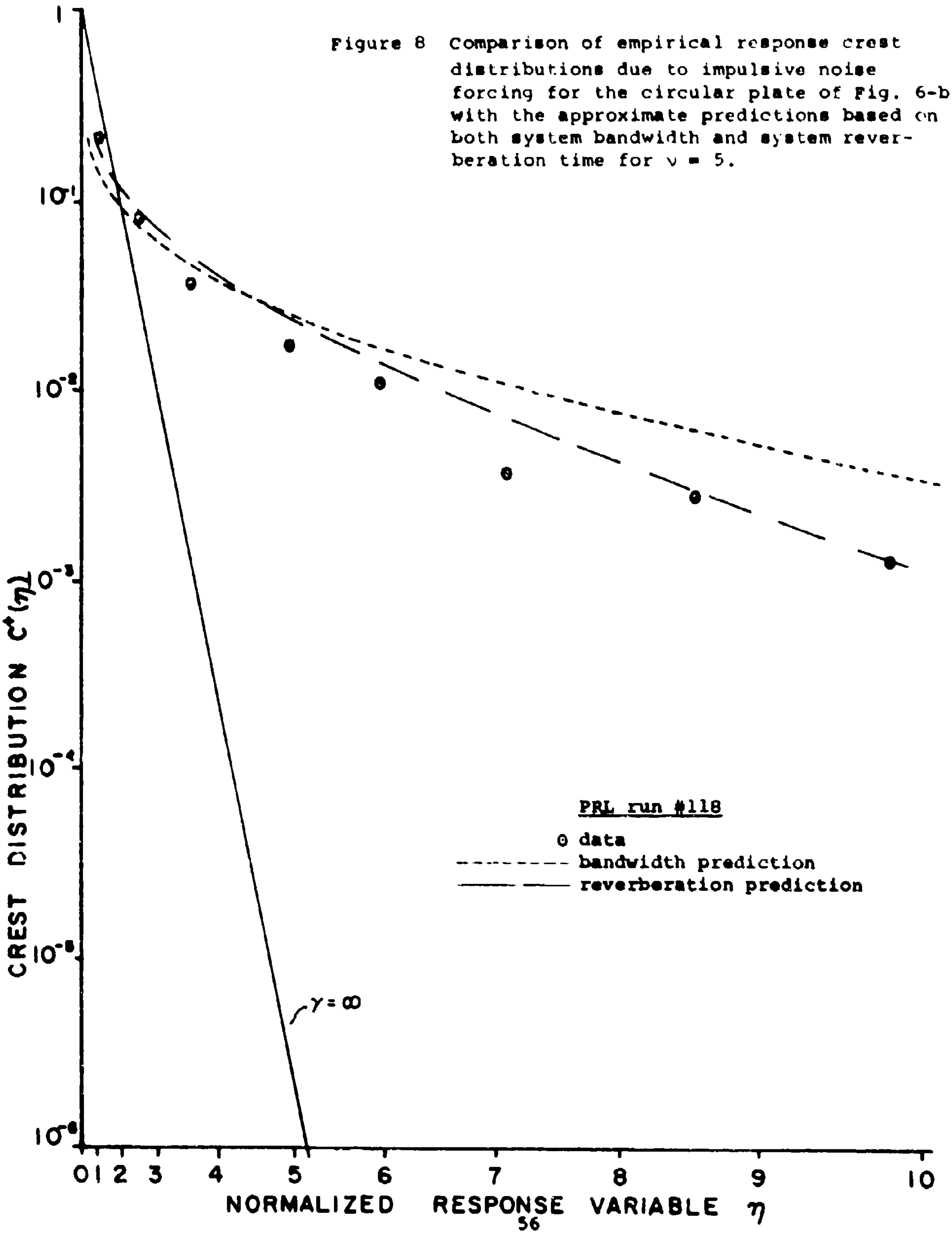


Figure 7 Comparison of empirical response crest distributions due to impulsive noise forcing for the circular plate of Fig. 6-b with the approximate predictions based on both system bandwidth and system reverberation time for $\nu = 15$.

Figure 8 Comparison of empirical response crest distributions due to impulsive noise forcing for the circular plate of Fig. 6-b with the approximate predictions based on both system bandwidth and system reverberation time for $\nu = 5$.



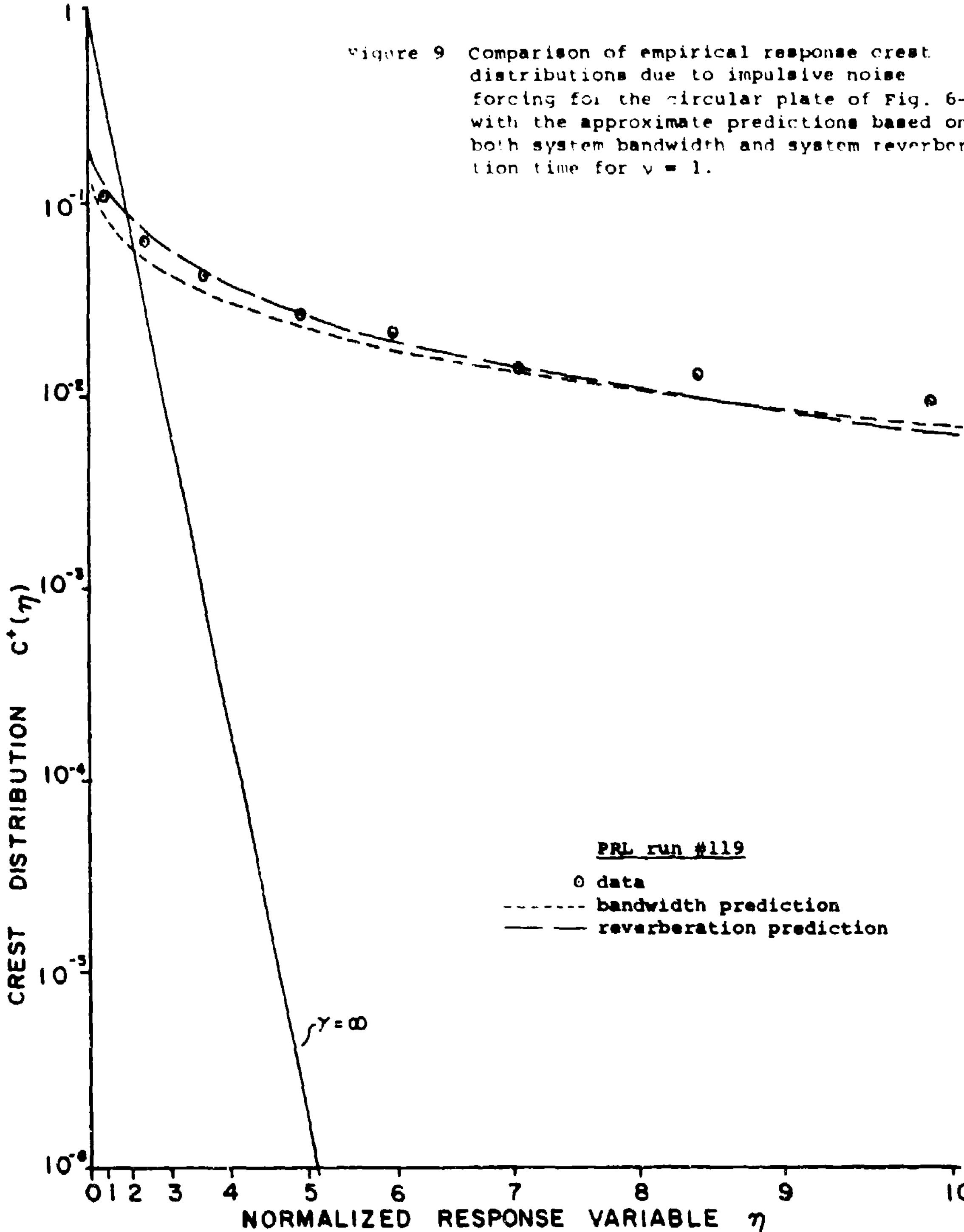


Figure 9 Comparison of empirical response crest distributions due to impulsive noise forcing for the circular plate of Fig. 6-b with the approximate predictions based on both system bandwidth and system reverberation time for $\nu = 1$.

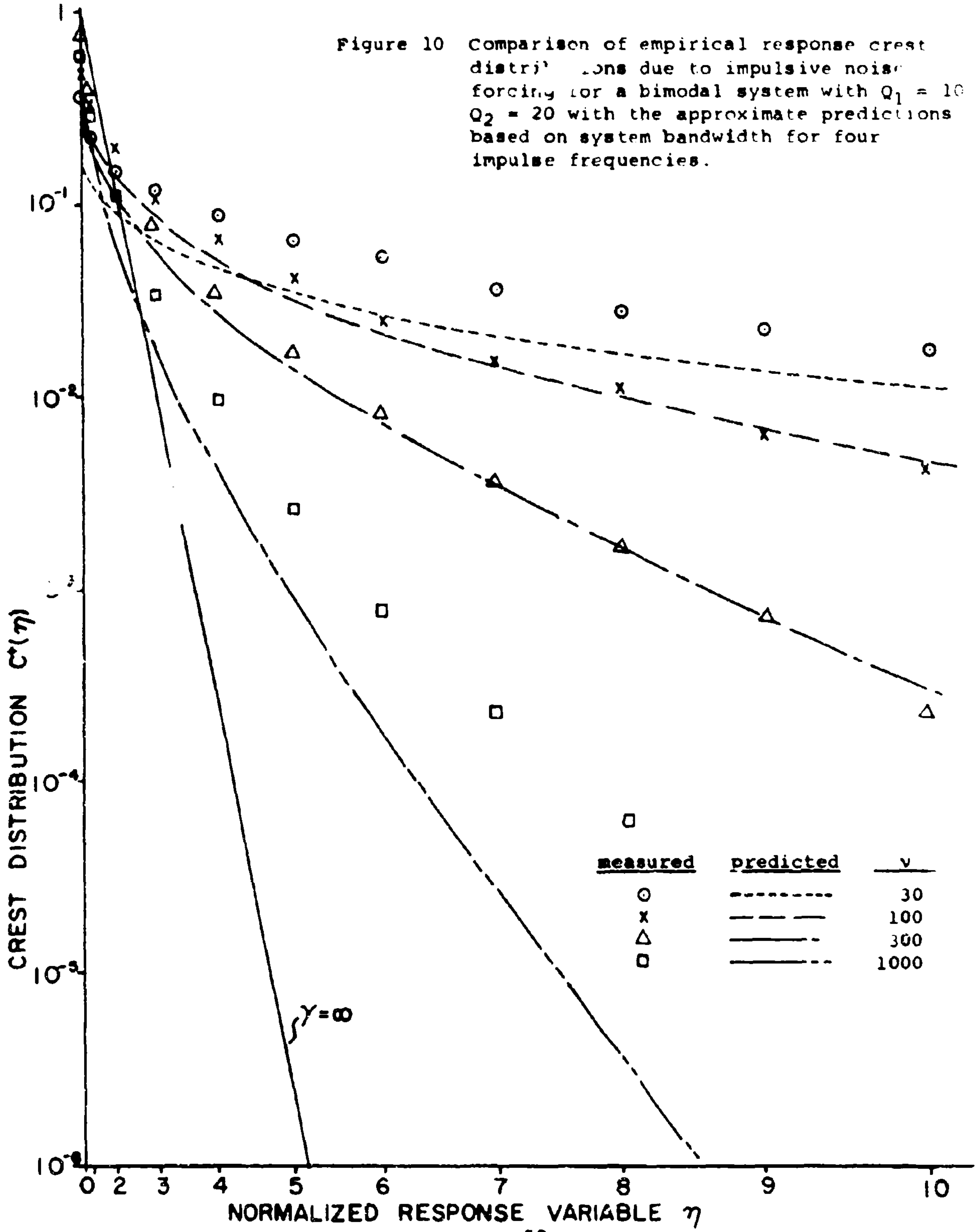


Figure 10 Comparison of empirical response crest distributions due to impulsive noise forcing for a bimodal system with $Q_1 = 10$ $Q_2 = 20$ with the approximate predictions based on system bandwidth for four impulse frequencies.

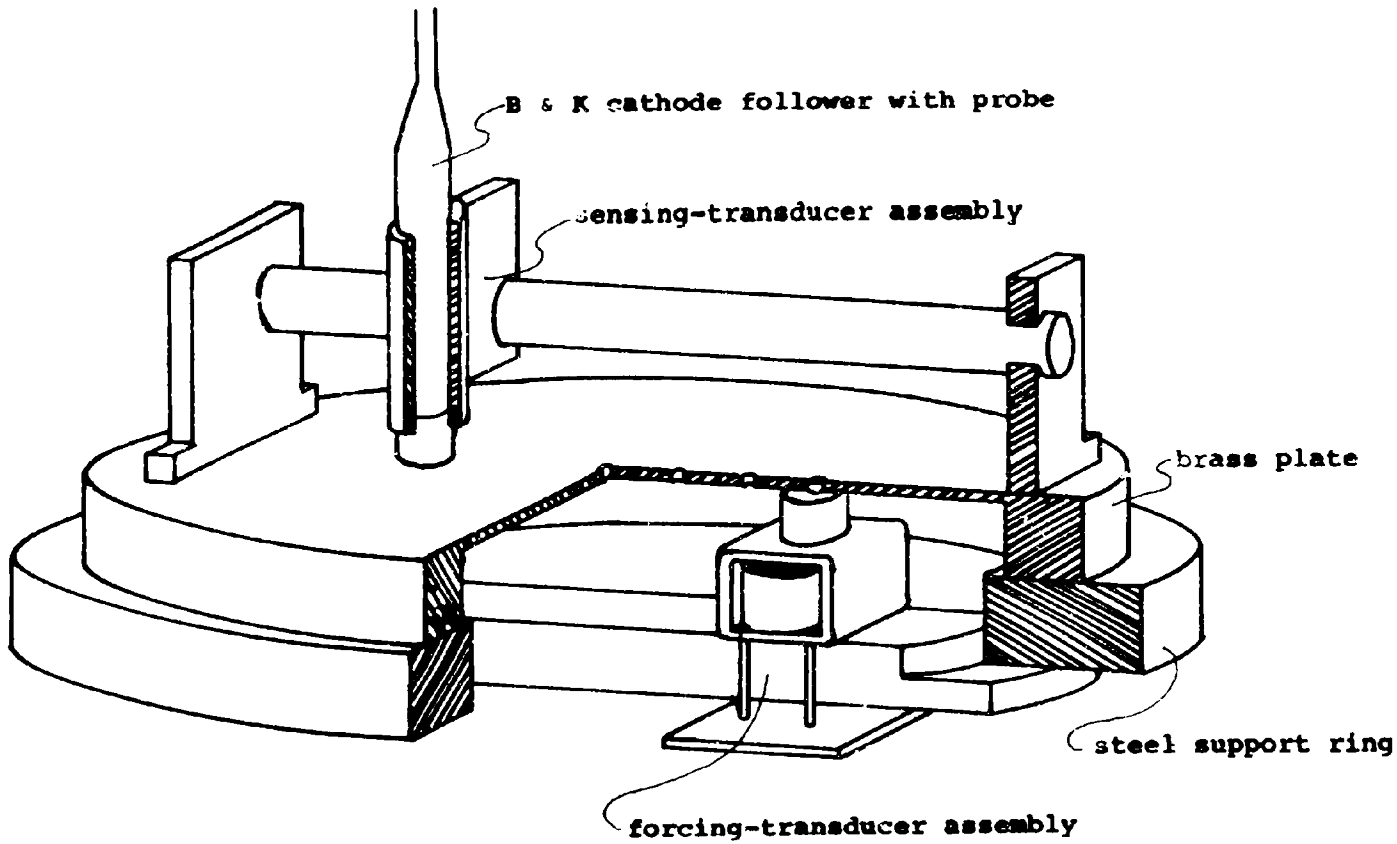


Figure 11 Illustration of the clamped-circular plate construction indicating placement of the electromagnetic driver and the electrostatic sensing probe mountings.

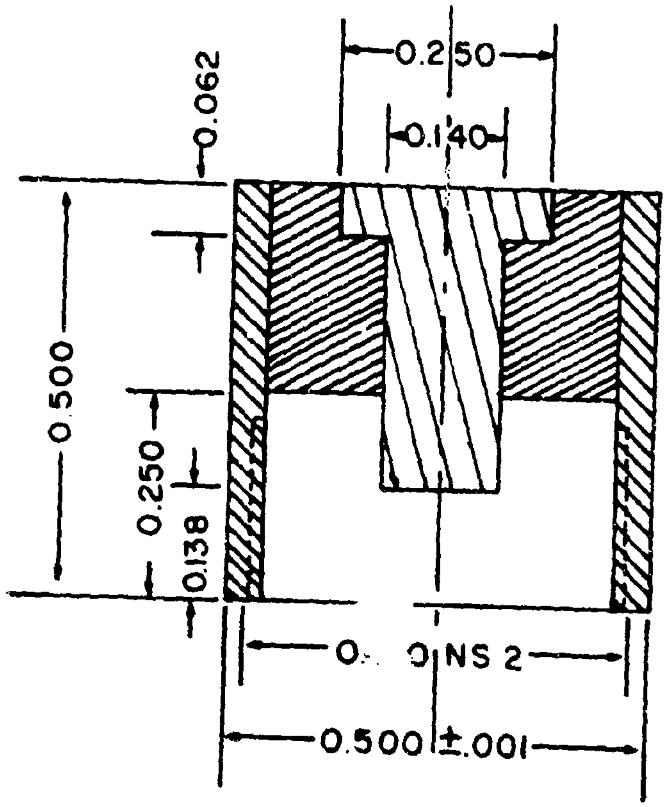


Figure 12 Details of the electrostatic sensing probe cap designed to facilitate mounting on a Bruel and Kjaer cathode follower unit.

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13. ABSTRACT

The current analytical models of impulsive noise and system response thereto are found numerically intractable, indicating a need for engineering approximations. One, based on finding an "equivalent" unimodal system having approximately the same response statistics--here the crest statistics--is examined. It is found that fair agreement obtains, making it possible to use two reasonably accessible parameters (and) to characterize any linear multimodal system response and thus predict its crest statistics systematically, provided that "standard" curves of unimodal response crest statistics are available for statistically identical forcing. Limitations of this promising approach are explored, encouraging its use in reliability prediction pending further studies. Also, the need for further studies of impulsive noise response crest distribution "tails" is noted for prediction of overload and wear-out reliability on the basis of wear-dependent failure rate concepts, since extrapolation of empirical data into the "tail" (here done on the basis of a Weibull crest fit) is increasingly necessary the more "impulsive" the response is (i.e., the more the parameter is less than unity).

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